THE PROBLEM OF TWO ACES

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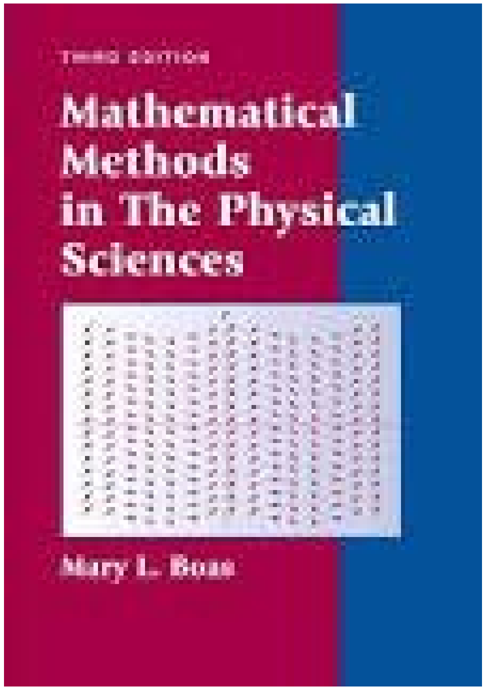
Chapter 15: Probability & Statistics
Section 4 Problem 8 on page 743



Two cards are drawn from a shuffled deck. What is the probability that both are aces? If you know that at least one is an ace, what is the probability that both are aces? If you know that one is the ace of spades, what is the probability that both are aces?

Suggested rewording of the problem

(i) You are dealt two cards face down from a shuffled deck. What is the probability you got two aces?

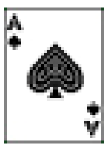
(ii) A computer does the dealing and checks the cards before laying them on the table. If neither card is an ace, it replaces both cards in the deck, reshuffles the deck, and repeats the process. As soon as the hand contains at least one ace, the computer lays the hand face down on the table. Now what is the probability you got two aces?

(iii) Same as the second case, but this time the computer makes sure your hand contains the ace of spades before laying it down on the table. Now what is the probability?

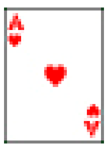
Method of solution

$$\text{probability of getting what you want} = \frac{\text{number of ways to get what you want}}{\text{number of ways to get anything possible}} = \frac{N_{\text{win}}}{N_{\text{win}} + N_{\text{lose}}}$$

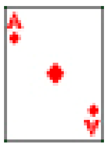
let $P(N,n)$ and $C(N,n)$

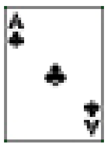


denote number of permutations and combinations of N objects taken n at a time



Initially assume the two cards are not ordered (in either spatial position or time of dealing).





C = ace of clubs, D = ace of diamonds,
H = ace of hearts, S = ace of spades

(i) You are dealt two cards face down from a shuffled deck. What is the probability you got two aces?

quick answer: probability of getting one ace $\times$ probability of getting another ace

$$\frac{4}{52}\times\frac{3}{51}=\boxed{\frac{1}{221}}$$

formal method: $N_{\text{win}} = C(4,2) = 6$ namely $\{\text{CD}, \text{CH}, \text{CS}, \text{DH}, \text{DS}, \text{HS}\}$

$$N_{\text{lose}} = C(48,2) + C(4,1)C(48,1) = 1320$$

$C(48,2)$ = number of ways of getting two non-aces

$C(4,1)C(48,1)$ = number of ways of getting one ace and one non-ace

$$\therefore \text{probability} = \frac{6}{6+1320} = \boxed{\frac{1}{221}}$$

(ii) A computer does the dealing and checks the cards before laying them on the table. If neither card is an ace, it replaces both cards in the deck, reshuffles the deck, and repeats the process. As soon as the hand contains at least one ace, the computer lays the hand face down on the table. Now what is the probability you got two aces?

N_{win} still equals 6

but now $N_{\text{lose}} = C(4,1)C(48,1) = 192$

because the possibility of two non-aces is excluded

$$\therefore \text{probability} = \frac{6}{6+192} = \boxed{\frac{1}{33}}$$

(which is almost seven times bigger than before)

(iii) This time the computer makes sure your hand contains the ace of spades before laying it down on the table. What is the probability you got two aces?

Is the answer the same as in the previous case?

No:

$N_{\text{win}} = C(3,1) = 3$ namely $\{\text{CS}, \text{DS}, \text{HS}\}$

because if we already have the ace of spades there are only 3 ways of getting another ace

$N_{\text{lose}} = C(48,1) = 48$

there are 48 ways of getting a non-ace instead

$$\therefore \text{probability} = \frac{3}{3+48} = \boxed{\frac{1}{17}}$$

Now just hold on a minute…

In the third case, I’ve already got the ace of spades.
There are 51 cards left in the deck, and 3 of them are aces.
So my odds of winning are 3/51 = 1/17.
That makes sense.

But why does it matter that the known ace was spades?
Shouldn’t this also be the answer in the second case,
when I started out with any ace?

Let's distinguish the two cards as the right and left ones.

Count possible hands that contain at least one ace:
There are 51 distinct hands with S (ace of spades) on the left,
51 more with C on the left, and so on for a total of $4 \times 51 = 204$ hands.

Next, there are 51 new hands that have S on the right,
except we have to subtract 3 hands we already counted above
because they had aces on the left: HS, CS, and DS.
Likewise for the other aces on the right, for a total of $4 \times 48 = 192$ hands.

So there's a grand total of 396 ordered hands that contain at least 1 ace.
Of them, $4 \times 3 = 12$ have aces both on the left and right.

This confirms that our winning odds are 12/396=1/33 for the 2nd case.

Look at it this way:

There are *twice* as many ways to get ANY two aces
as to get the ace of spades plus one other ace.

But there are *four* times as many ways to get any ace plus any non-ace
as to get the ace of spades plus any non-ace.
And this number is much larger than the preceding number,
so it's roughly equal to the total number of possibilities.

Thus your odds are roughly halved (from 1/17 to 1/33)
if you merely know you have an ace but don't know its suit.

Why teach this to physics students?

The more relevant information you have,
the more your odds go up.

ENTROPY is a measure of how much relevant
information one has about a situation.

Entropy might be less mysterious
if probability ideas were better understood.

In that case, can you give me some other similar questions to practice on?

The Greens have two children. What is the probability that both are boys? How about if you know that at least one of their kids is a boy? How about if you know that their oldest child is a boy? ("Boy or Girl Paradox" on wikipedia)

You're on a game show and are given a choice of 3 doors, behind one of which is a car and behind the other two are goats. You pick a door. The host is required to then open another of the doors behind which there is a goat. Would it be to your advantage to switch from your original choice to the other unopened door? ("Monty Hall Problem" on wikipedia)

On second thought, can I go? My head hurts!