Why Don't We Teach the Helmholtz Theorem

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Spirit of the Talk

- My question isn't philosophical
  - I'm not Socrates and none of you are Euthyphro
  - It's meant to open or reopen a dialog
- Full Disclosure
  - I'm not an educator
  - I'm not an expert in Electromagnetism
  - I was (and perhaps am) a student who finds the way we commonly teach E&M inadequate
- My aim is to make E&M more dynamic both literally and pedagogically

Outline

- Current textbook approach [1-6] obscures the unity of Maxwell's equations
  - Starts with static fields
  - Builds on time dependence
  - 'Sneaks' the displacement current in
- A better approach is that of Solymar [7]
  - Start with Maxwell's equations as given
  - Interrogate them to get specific results of interest
  - No unlearning (e.g. $\nabla \times \vec{E} \neq 0$)
- Even better approach is a modification of Solymar's approach using the Helmholtz theorem

Helmholtz Theorem [8-10]

- Start with the resolution of the delta function in spherical coordinates (Gauss's law in disguise)

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}')$$

- Decompose a function against the delta as

$$\vec{F}(\vec{r}) = \int_V d^3r'\,\delta(\vec{r}-\vec{r}')\vec{F}(\vec{r}') = -\frac{\nabla^2}{4\pi}\int_V d^3r'\,\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

where V is a bounded volume containing $\vec{r}'$

- Use the identity laplacian() = grad(div())-curl(curl()) and allow V to be all space

$$\vec{F}(\vec{r}) = -\nabla \mathrm{U}(\vec{r}) + \nabla\times\vec{W}(\vec{r})$$

where $U(\vec{r}) = \frac{1}{4\pi}\int_V d^3r'\,\frac{\nabla'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ and $\vec{W}(\vec{r}) = \frac{1}{4\pi}\int_V d^3r'\,\frac{\nabla'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

Pedagogical Advantages

1. From the structure of the theorem the divergence and curl is sufficient to specify the field
   a) Justifies why Maxwell's equations deal only with div() and curl() (no symmetric, traceless derivative)
   b) Contrasts with 'ordinary functions' that need a constant of integration

2. Early exposure to the delta-function
   a) Introduces a Green function early on in a meaningful way
   b) Breaks what I call the 'tyranny of analytic functions' – Penrose's 'message driven economy' [11]

3. Derivation of Coulomb's and Biot-Savart's laws
   a) These 'laws' fall out simply (see following slide)
   b) No special symmetry, no idealized cases, no hand-waving

Maxwell's Equations (mks units)

Basic Law	General Form	Static Form
Gauss	$\nabla \cdot \vec{E}(\vec{r},t) = \rho(\vec{r},t)/\epsilon_0$	$\nabla \cdot \vec{E}(\vec{r}) = \rho(\vec{r})/\epsilon_0$
Monopole	$\nabla \cdot \vec{B}(\vec{r},t) = 0$	$\nabla \cdot \vec{B}(\vec{r}) = 0$
Faraday	$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$	$\nabla \times \vec{E}(\vec{r}) = 0$
Ampere	$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t) + \epsilon_0\mu_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$	$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$

- Coulomb's law: (Gauss and Faraday)

$$U(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \vec{W}(\vec{r}) = 0 \qquad \vec{E}(\vec{r}) = -\nabla U(\vec{r})$$

- Biot-Savart's law: (Monopole and Ampere and $\nabla \times (\phi\vec{F}) = (\nabla\phi)\times\vec{F} + \phi\nabla\times\vec{F}$)

$$U(\vec{r}) = 0 \qquad \vec{W}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V d^3r' \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \vec{B}(\vec{r}) = \nabla \times \vec{W}(\vec{r})$$

Critique

This modification of Solymar's approach is not without difficulties

1. Boundary conditions on the integrals and fields
   a) If V is kept bounded there are some additional terms involving integrals of normal components on the boundary ∂V
   b) If V is all space then div and curl of the field must vanish faster than r^{-1}
   c) Obvious mathematical but subtle physical implications on the sources (cause and effect)
   d) Boundary conditions in the classroom always seem a bit tricky

2. Helmholtz theorem seems to only be valid for static fields
   a) More complicated form involving retarded time is advocated that allows for the derivations of Faraday and Ampere [12,13]
   b) Retarded time is an advanced concept
   c) This point may not be universally agreed upon (Jefimenko *seems* to me to apply Helmholtz theorem universally in [14])

THANKS FOR YOUR ATTENTION

References

- [1]: Physics, Parts 1& 2 Combined – Halliday & Resnick
- [2]: Foundations of Electromagnetic Theory – Reitz, Milford, & Christie
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- [4]: Electromagnetic Theory – Frankl
- [5]: Electromagnetism – Slater & Frank
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- [7]: Lectures on Electromagnetic Theory – Solymar
- [8]: Mathematical Methods for Physicists – Afken
- [9]: Miller, Am. J. Phys 52, 948 (1984)
- [10]: Mirman, Am. J. Phys 33, 503 (1965)
- [11]: The Road to Reality – Penrose
- [12]: Davis, Am. J. Phys 74, 72 (2006)
- [13]: Hera, Am. J. Phys 74, 743 (2006)
- [14]: Causality, Electromagnetic Induction, and Gravitation - Jefimenko