Off-Center Elastic Collisions Between Two Smooth Pucks

Carl E. Mungan, Physics Department
U.S. Naval Academy, Annapolis MD

Chesapeake Section of AAPT
Piedmont Virginia Community College, Charlottesville

28 April 2018

GOAL

Analyze a general 2D elastic collision. (Introductory texts usually only treat special cases, such as where at least one of the final scattering angles is known or where the two objects are identical.)

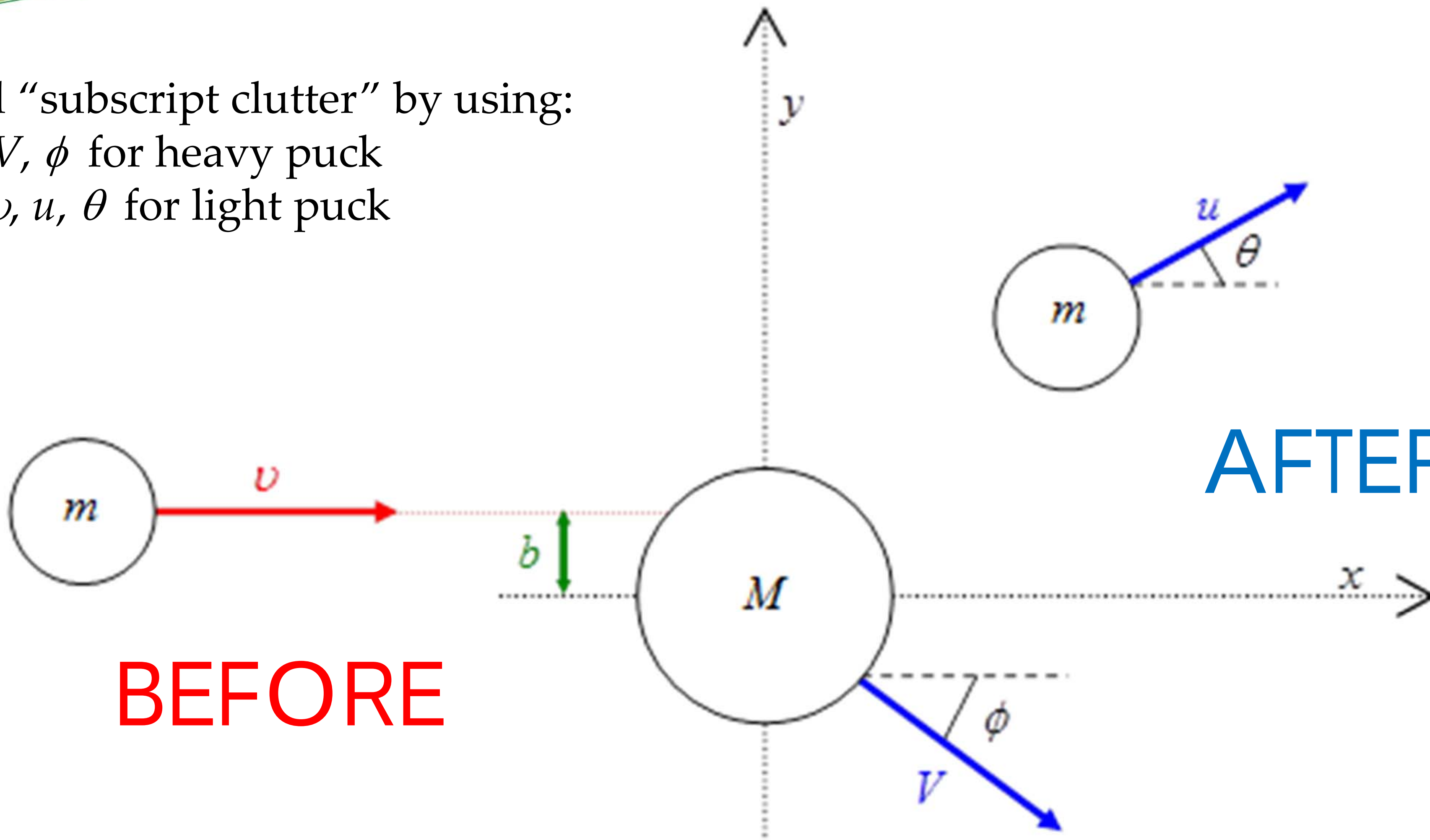
For simplicity, choose a reference frame in which one object is initially at rest. Of particular interest is finding a relationship between the two scattering angles. That would generalize the familiar result that the two angles add up to 90° for equal-mass objects such as pool balls.

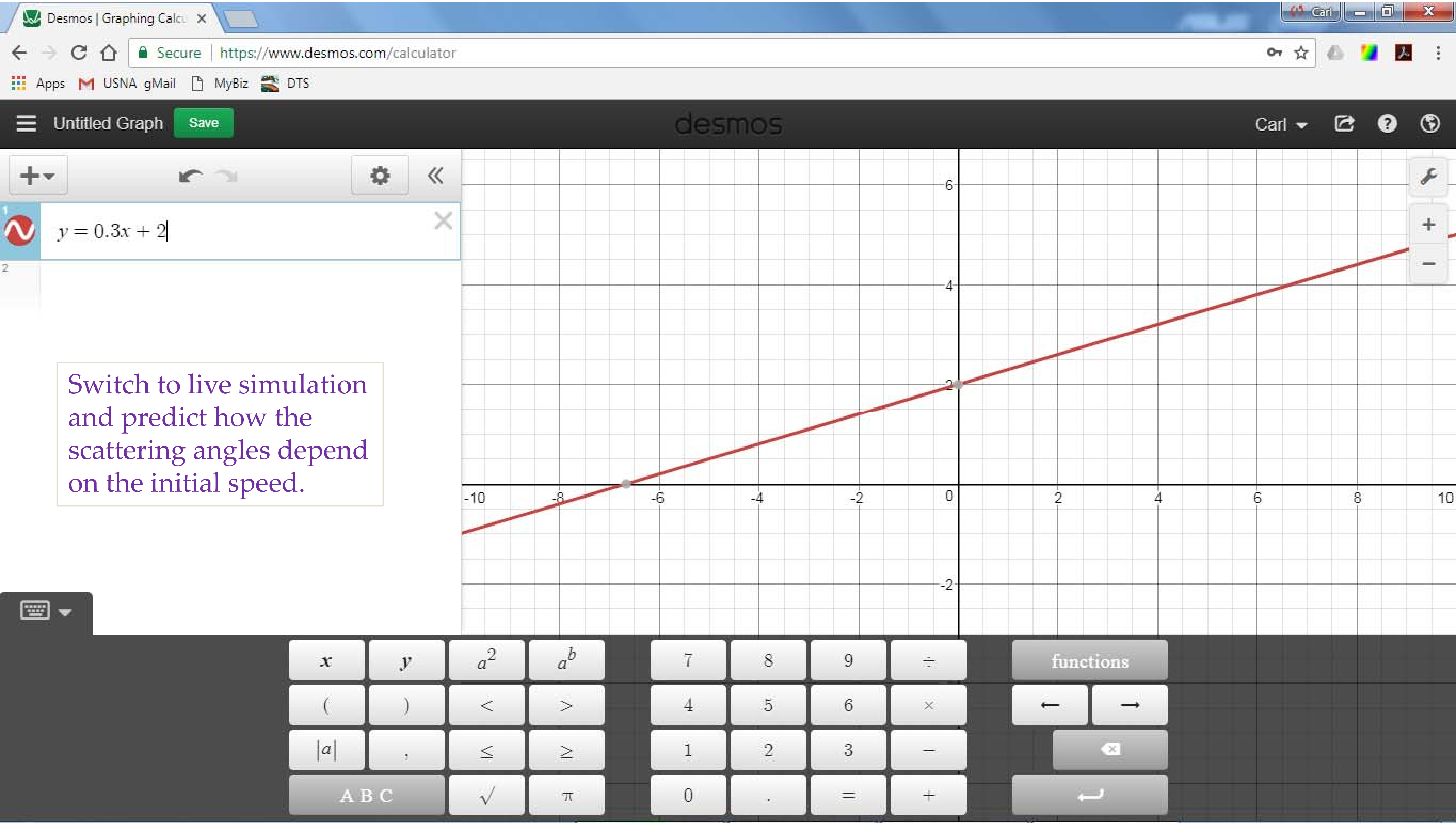
SETUP

Any off-center collision is two-dimensional. Choose the reference frame so the heavy puck of mass M is initially at rest at the origin. Choose the x-axis to be parallel to the initial velocity of the light puck of mass m. Choose the y-axis so the impact parameter b is positive. Assume the pucks float on a horizontal air table. For smooth pucks, there is no friction to start them rotating about their centers.

avoid "subscript clutter" by using:

- M, V, ϕ for heavy puck
- m, υ, u, θ for light puck





Switch to live simulation and predict how the scattering angles depend on the initial speed.

ANALYTIC SOLUTION

conserve *x* momentum: $$m\upsilon = mu\cos\theta + MV\cos\phi$$

conserve *y* momentum: $$mu\sin\theta = MV\sin\phi$$

conserve kinetic energy: $$\frac{1}{2}m\upsilon^2 = \frac{1}{2}mu^2 + \frac{1}{2}MV^2$$

Relation between the scattering angles

Eliminate u, υ, and V between these 3 equations to get:

$$\tan\theta = \frac{\sin 2\phi}{m/M - \cos 2\phi}$$

<u>Example</u> If $m = M$ then the double-angle formulas imply the two angles are complementary:

$$\tan\theta = \cot\phi$$

(try $r = 1$ in Desmos)

Other special cases

(for mass ratio $r = m/M$)

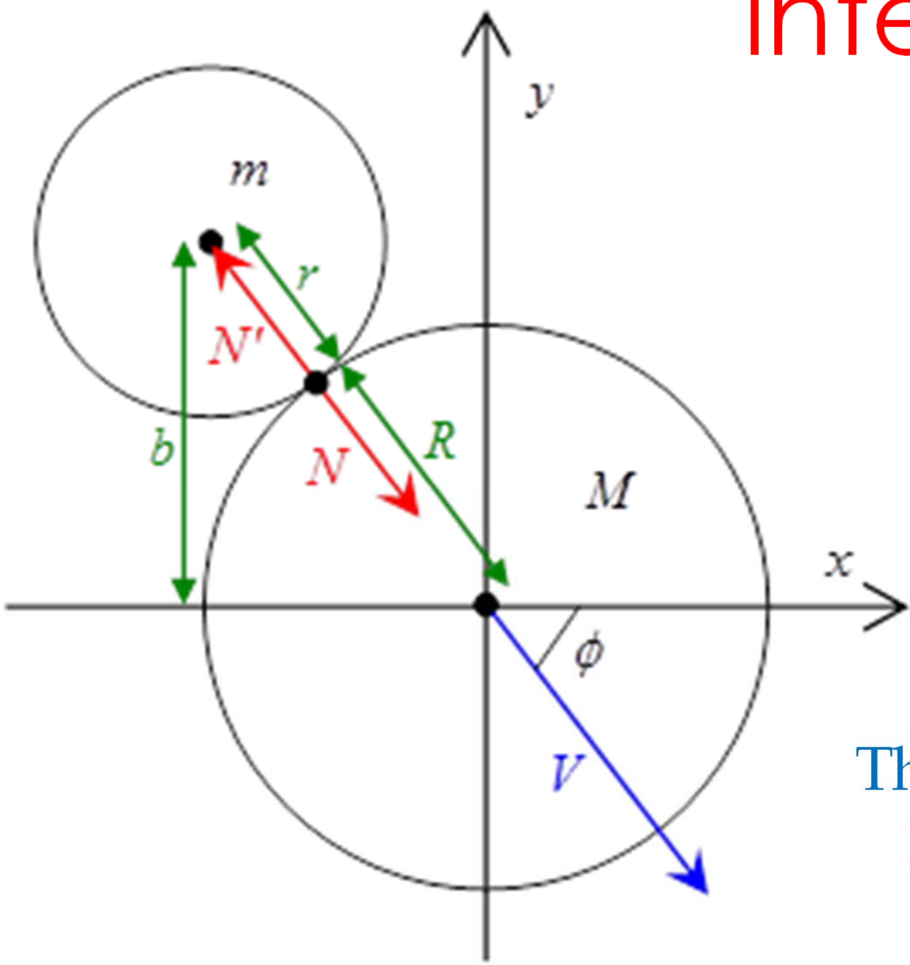
$$\text{A:}\quad \theta = 90° \quad \Rightarrow \quad \cos 2\phi = r$$

$$\text{B:}\quad \phi = 45° \quad \Rightarrow \quad \cot\theta = r$$

$$\text{C:}\quad \theta = \phi \quad \Rightarrow \quad 2\cos\phi = \sqrt{1+r}$$

$$\text{D:}\quad \theta = 2\phi \quad \Rightarrow \quad 2\cos\theta = r$$

Interaction forces



b = impact parameter

D = sum of radii of pucks

(choose radius 1 for the light puck and 2 for the heavy puck in arbitrary units)

The normal force N pushes the heavy puck in direction ϕ.

Completion of the solution

$$\boxed{\sin\phi = \frac{b}{D}}$$

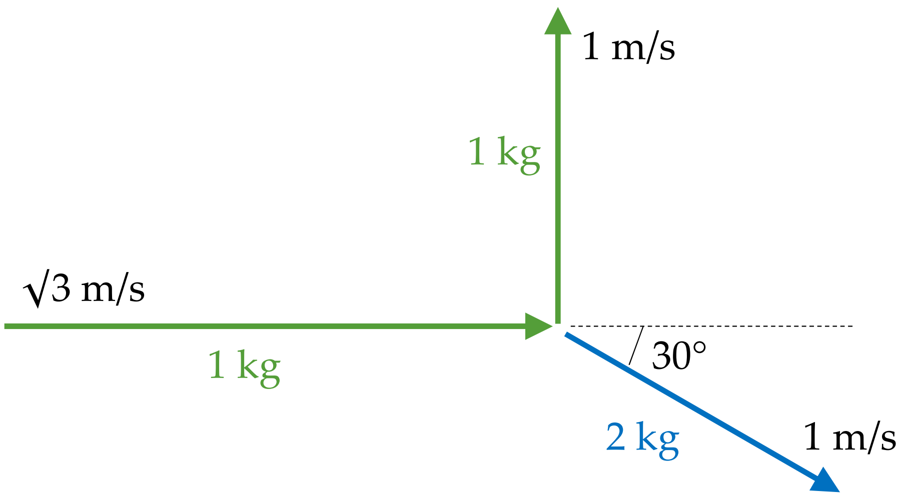
$\therefore$ scattering angles depend on the fractional impact parameter and mass ratio but not on the incident speed of the light puck

final speeds: $$\frac{V}{\upsilon} = \frac{r\sin\theta}{\sin(\theta+\phi)} \quad \text{and} \quad \frac{u}{\upsilon} = \frac{\sin\phi}{\sin(\theta+\phi)}$$

Example for case A

$$\theta = 90^\circ \text{ when } \cos 2\phi = r$$

$$r = \frac{1}{2} \Rightarrow \begin{cases} \phi = 30^\circ \\ b = \frac{3}{2} \end{cases}$$

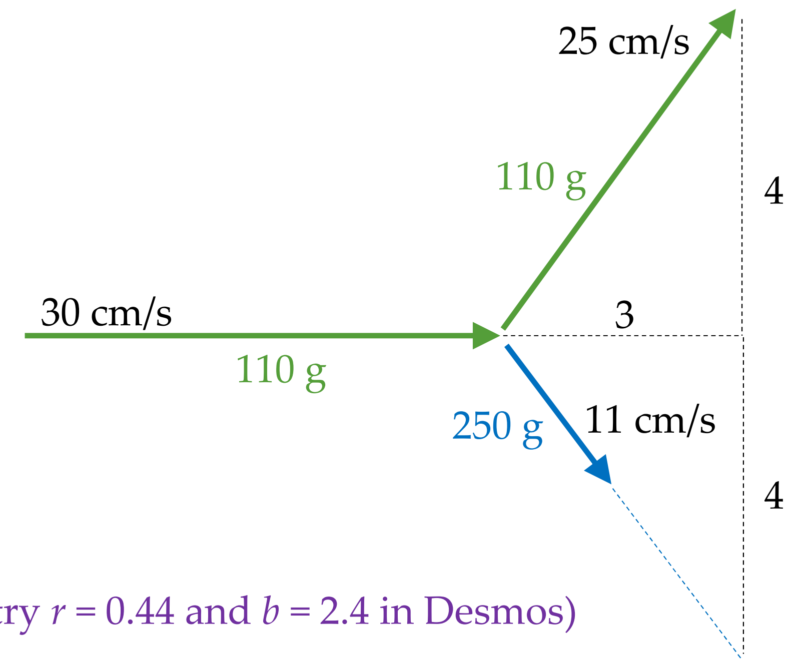


(try $r = 0.5$ and $b = 1.5$ in Desmos)

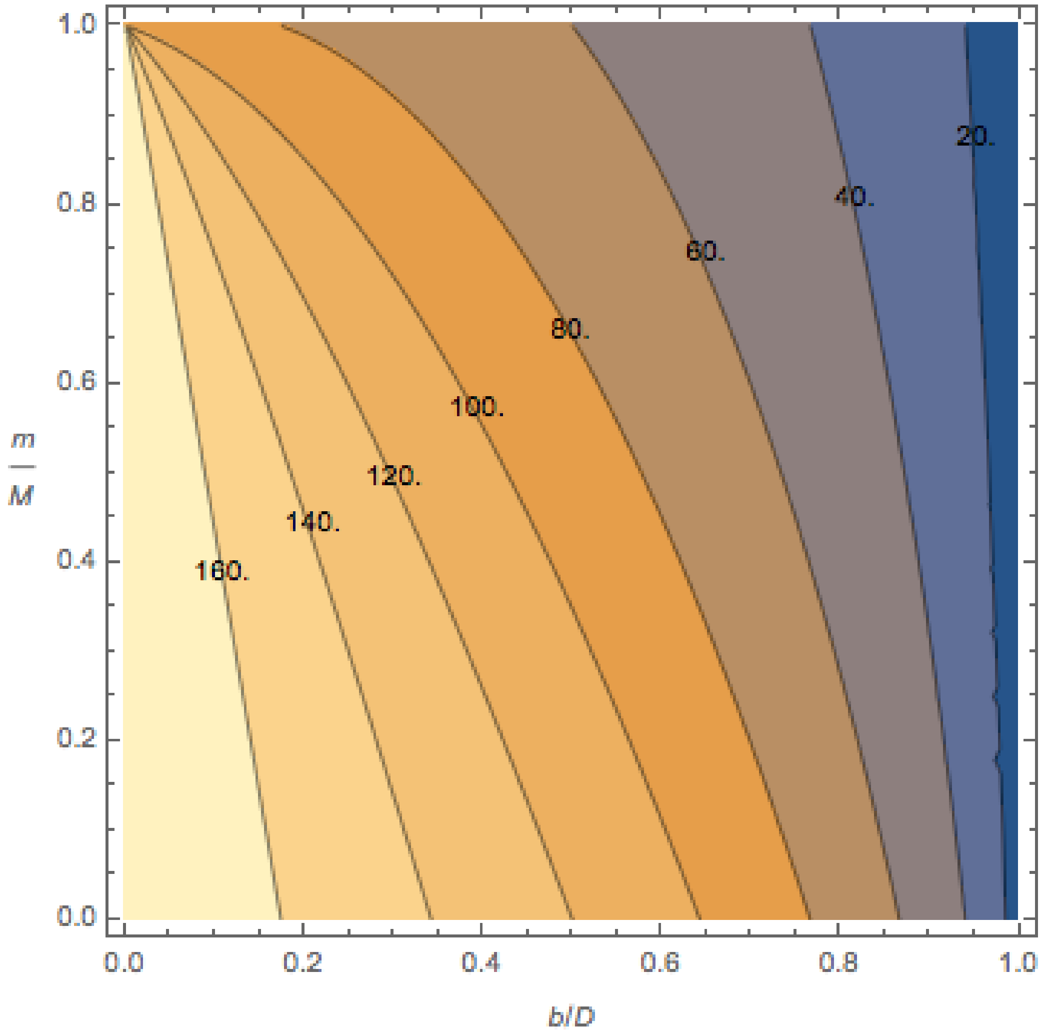
Example for case C

$$\theta = \phi \text{ when } 2\cos\phi = \sqrt{1+r}$$

$$r = 0.44 \Rightarrow \begin{cases} 3-4-5 \\ \text{triangle} \\ \\ b = 2.4 \end{cases}$$



(try $r = 0.44$ and $b = 2.4$ in Desmos)



Contour plot of the scattering angle (in degrees) of the light puck as a function of the fractional impact parameter and the mass ratio. The scattering is exactly backward ($\theta = 180°$) along the left edge of the graph for an on-center collision ($b = 0$) and is exactly forward ($\theta = 0°$) along the right edge of the plot for a grazing collision ($b = D$).

REFERENCE

Carl E. Mungan and Trevor C. Lipscombe, "Oblique elastic collisions of two smooth round objects," *European Journal of Physics* **39**, 045002 (July 2018)

available on my webpage at usna.edu/Users/physics/mungan