Chesapeake Section of the American Association of Physics Teachers

VERTICAL MOTION OF A BALL SUBJECT TO AIR DRAG

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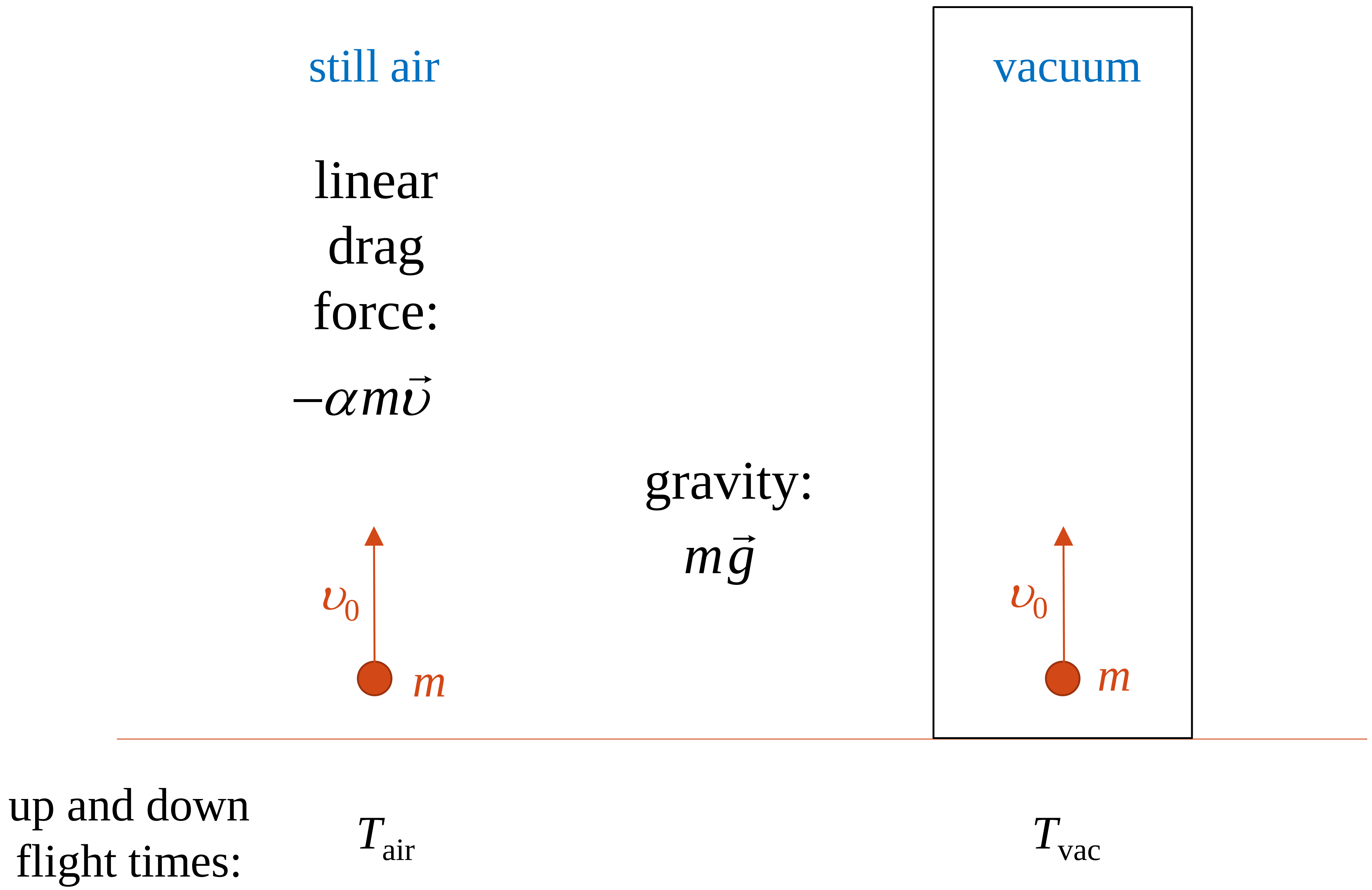

still air
linear drag force:
$-\alpha m\vec{\upsilon}$
gravity:
$m\vec{g}$
vacuum
υ_0
m
υ_0
m
up and down flight times:
T_{air}
T_{vac}

Compare flight times T_{air} and T_{vac}:

1. T_{air} is longer
2. T_{vac} is longer
3. both are equal
4. it depends on υ_0, m, α, or other factors

Discuss with your neighbors!

Answer is $T_{vac} > T_{air}$:

Consider the limit of weak gravity but high atmospheric density. In vacuum the peak height will be large and so will be the flight time. In air the high density will cause the ball to stop quickly. But drag is low when the ball begins to fall back and so it will still pick up downward speed without undue delay.

Elegant proof using impulse-momentum theorem:

upwards positive: $\dfrac{d\upsilon_y}{dt} = -g - \alpha \dfrac{dy}{dt}$

integrate over t: $\upsilon_{\text{f}y} - \upsilon_{0y} = -gT - \alpha\left(y_{\text{f}} - y_0\right)$

returns to ground with $y_{\text{f}} = y_0$ and $\upsilon_{\text{f}y} = -\upsilon_{\text{f}}$ final speed

$\therefore T_{\text{air}} = \dfrac{\upsilon_{\text{f}} + \upsilon_0}{g}$ but $\upsilon_{\text{f}} = \upsilon_0$ in vacuum so $T_{\text{vac}} = \dfrac{\upsilon_0 + \upsilon_0}{g}$

since $\upsilon_{\text{f}} < \upsilon_0$ in air, we conclude $T_{\text{vac}} > T_{\text{air}}$

Drag has zero net impulse over the entire trip!

Detailed solution using Newton's second law:

in vacuum $T_{\text{vac}} = 2T_{1/2} = 2\upsilon_0 / g$

in air let ground be height 0 and peak be height H

it takes less time T_{up} to go from 0 to H than time T_{down} from H back to 0

(because mechanical energy is continuously lost to drag)

and so it suffices to prove $T_{1/2} > T_{\text{down}}$

upward trip: $$\frac{d\upsilon}{dt} = -g - \alpha\upsilon \quad \Rightarrow \quad T_{\text{up}} = -\int_{\upsilon_0}^{0} \frac{d\upsilon}{g + \alpha\upsilon} = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha\upsilon_0}{g}\right)$$

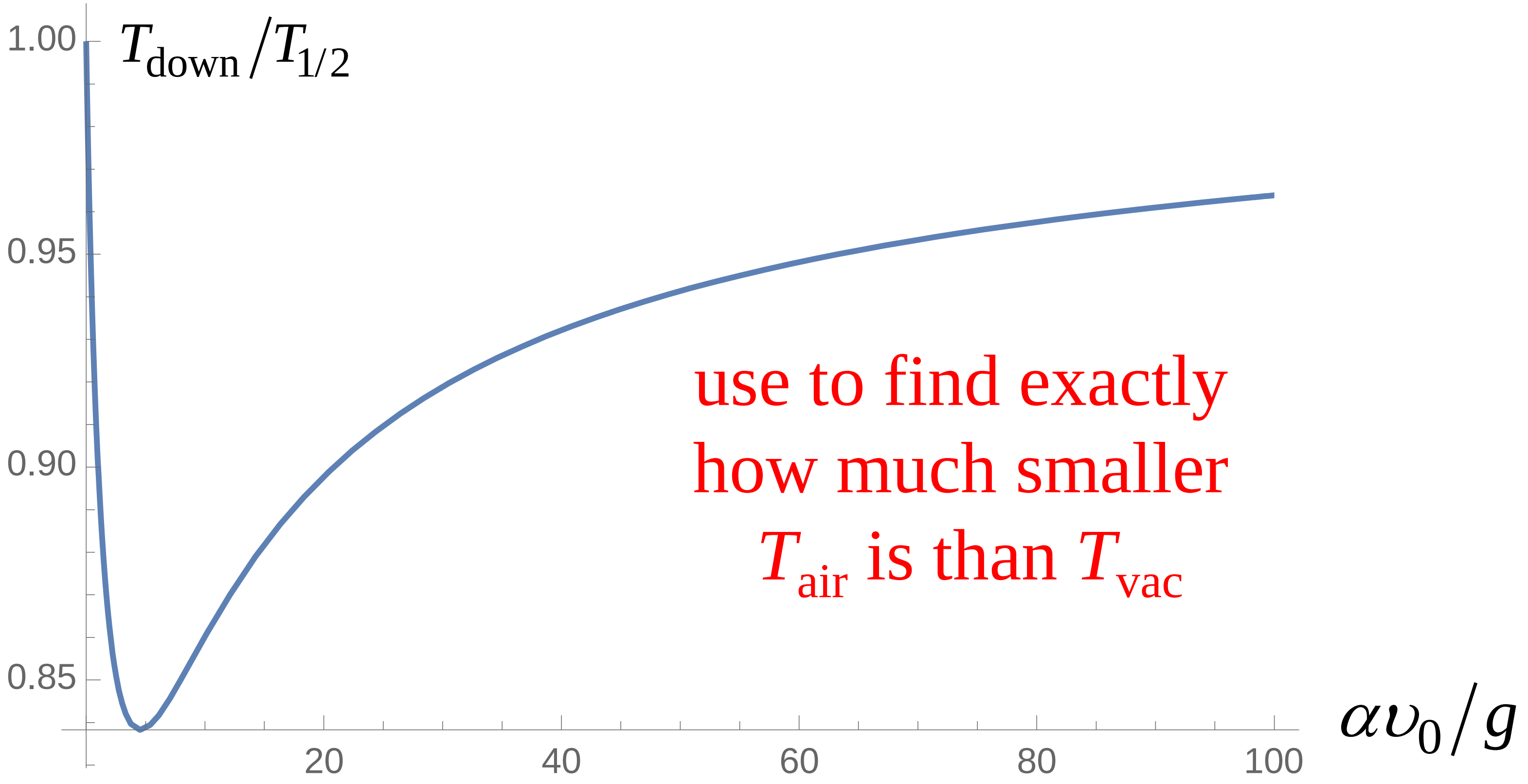
integrate again $$\Rightarrow \quad H = \frac{gT_{1/2}}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \alpha T_{1/2}\right)$$

downward trip: $\frac{d\upsilon}{dt} = g - \alpha\upsilon$

similar steps $\Rightarrow$ $H = \frac{gT_{\text{down}}}{\alpha} - \frac{g}{\alpha^2}\left(1 - e^{-\alpha T_{\text{down}}}\right)$

equate expressions for H $\Rightarrow$ $\left(1+\alpha T_{1/2}\right) - \ln\left(1+\alpha T_{1/2}\right) = \alpha T_{\text{down}} + e^{-\alpha T_{\text{down}}}$

which can be solved in terms of the Lambert W function to get the graph:



More realistic drag force on a ball:

$$\text{Reynolds number } \text{Re} = \frac{\rho D \upsilon}{\eta}$$

$$\text{density of air } \rho = 1.21 \text{ kg/m}^3$$

$$\text{viscosity of air } \eta = 1.83 \times 10^{-5} \text{ Pa} \cdot \text{s}$$

$$\text{diameter of baseball } D = 7.4 \text{ cm}$$

$$\therefore \text{Re} > 1000 \text{ whenever } \upsilon > 0.2 \text{ m/s}$$

(true everywhere except within 2 mm of the peak height)

which implies the drag is *quadratic* not linear

with magnitude $F_D = mg(\upsilon / \upsilon_T)^2$ in terms of the terminal speed υ_T

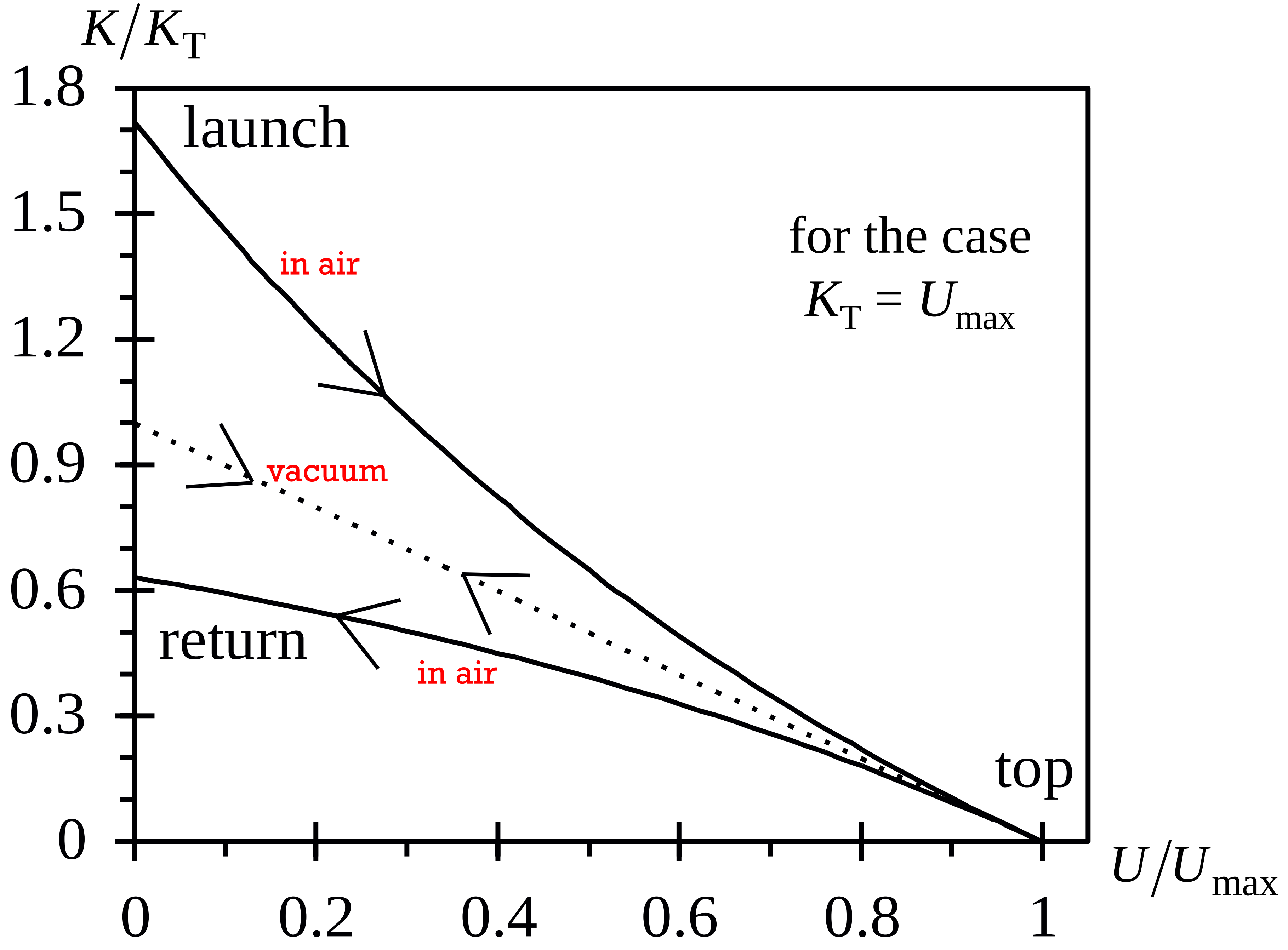
Mechanical energy is lost to drag:

upward trip: $dK + dU = \vec{F}_{\mathrm{D}} \cdot d\vec{r} = -mg\left(\frac{\upsilon}{\upsilon_{\mathrm{T}}}\right)^2 dy = -\frac{K}{K_{\mathrm{T}}} dU$

separate variables and integrate $\Rightarrow$ $H = \frac{\upsilon_{\mathrm{T}}^2}{2g} \ln\left(1 + \frac{\upsilon_0^2}{\upsilon_{\mathrm{T}}^2}\right)$

downward trip: $dK + dU = \frac{K}{K_{\mathrm{T}}} dU$ $\Rightarrow$ $H = -\frac{\upsilon_{\mathrm{T}}^2}{2g} \ln\left(1 - \frac{\upsilon_{\mathrm{f}}^2}{\upsilon_{\mathrm{T}}^2}\right)$

Define $K_{\mathrm{T}} = \frac{1}{2} m\upsilon_{\mathrm{T}}^2$ and $U_{\max} = mgH$ to get the normalized graph shown next.



K/K_T
1.8
1.5
1.2
0.9
0.6
0.3
0
launch
in air
vacuum
return
in air
top
for the case
K_T = U_max
0
0.2
0.4
0.6
0.8
1
U/U_max

Memorable sum of reciprocal kinetic energies

equate formulas for H $\Rightarrow$ $$\boxed{\frac{1}{K_f} = \frac{1}{K_0} + \frac{1}{K_T}}$$

which implies:

(1) the landing speed is always smaller than both the launch and terminal speeds

(2) we recover the drag-free result $\upsilon_f = \upsilon_0$ if $\upsilon_0 << \upsilon_T$

(3) the landing speed equals the terminal speed if $\upsilon_0 >> \upsilon_T$

QUESTIONS?

COMMENTS WELCOME BY EMAIL TO MUNGAN@USNA.EDU

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where you can find

- "Energy-based solution for the speed of a ball moving vertically with drag," Eur. J. Phys. **27**, 1141 (2006)
- "Vertically up and down motion with and without drag," White Papers (2019)