The Spin-Statistics Theorem: Just an Axiom or a Consequence of Relativity?

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Outline

- The Question
- The Topological approach
- The Linear Algebra approach
- Current state of the theorem
- How to teach it

Spin and Statistics in Quantum Mechanics

- Particles with half-integer spin are fermions
  - Obey the Pauli exclusion principle
  - No two particles in the same system can have the same set of quantum numbers

- Particles with integer spin are bosons
  - Do not obey the Pauli Exclusion principle
  - Phenomena such as superfluidity is possible

The Sign Difference between Bosons and Fermions

From Griffiths and Schroeter *Introduction to Quantum Mechanics*, 3rd ed, section 5.1.1:

- The wavefunction for a 2 particle system:

$$\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = A\left[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)\right];$$

- Bosons are symmetric under exchange and obey the plus sign
- Fermions are antisymmetric under exchange and obey the minus sign
  - Note wavefunctions of two identical particles in the same states cancel each other out; this is the Pauli Exclusion principle

- Accepted proofs connecting the sign to spin rely on relativistic quantum mechanics

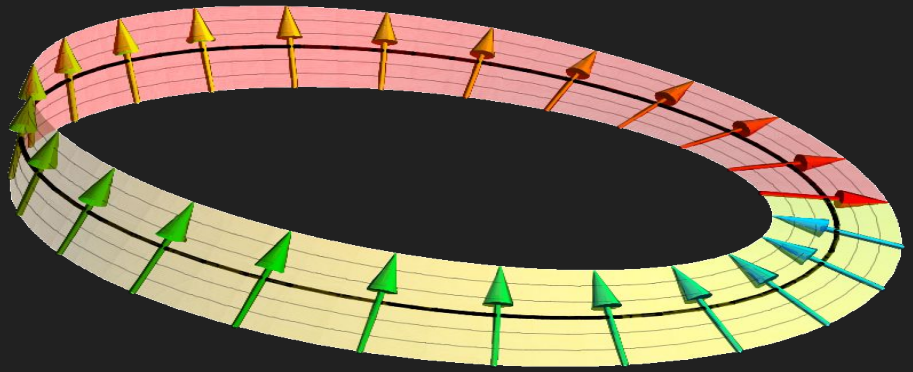
Neuenschwander’s Question

- Posted in the American Journal of Physics, 1994
- “Has anyone made any progress towards an ‘elementary’ argument for the spin-statistics theorem?”
- Subsequent responses seek to explain the origin of the minus sign in the 2-particle wavefunction for fermions

Gould’s 1995 Reply

- Suggests Feynman answered this question using topology: electrons act as spinors and change sign under rotation
- Offers the Philippine Wine Dance as a Demonstration
- Forester concurs and suggests a demonstration with rubber bands

To the right:
Visualization of a spinor
(notice how the arrow flips as it is translated across the Möbius strip)



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Flaws:

- Hilborn (1995): an analogy is not a proof
- Duck and Sudarshan (1998) : Feynman’s argument requires extraneous features and have extraneous consequences

Bacry's 1995 (and Broyles's 1976 and 2002) reply

- Shows that a system of spin ½ particles changes sign under rotation
- Says proof for spin 0 is trivial and a matrix swap sufficient for spin 1
    - What about spin 2 or other integers?

Flaws:

- Hilborn, Duck and Sudarshan: State invariance does not require wavefunction invariance
- Romer: replies to Broyles in 2002, notes Hilborn, Duck, and Sudarshan did read his paper, points out state-wavefunction invariance flaw

Timeline

- 1976-Broyles’s attempt
- 1994-Neuenschwander’s question
- 1995-Gould’s analogy to topology
- 1995-Bacry’s linear algebra attempt
- 1995-Hilborn’s critique of Gould and Bacry
- 1996-Forester’s concurrence with Gould
- 1998-Duck and Sudarshan’s refutation of the failed proofs
- 1999-Wightman’s review of Duck and Sudarshan’s book
- 2002-Romer’s refutation of Broyles

Current state of theorem

- Duck and Sudarshan (1998): outlines a proof by Schwinger that minimizes dependence on relativity but doesn’t eliminate it
  - To eliminate dependence, flavor symmetry must be taken as a postulate and not a result of special relativity
- Scammell and Sushkov (2015): theoretical paper in *Physical Review Letters* considering Bose condensation of spin ½ particles
  - Such condensates would have distinctive “smoking gun” features

Teaching the Spin-Statistics theorem

- Probably can’t do it at undergrad, but can mention it
- For theoreticians: mention mathematical basis
  - Magnetism (absent the discovery of monopoles) has been accepted as a consequence of relativity. Can we classify the spin statistics theorem the same way?
  - Any proof relativity may NOT be necessary?
- For experimentalists: mention experiments into the behaviors of bosons and fermions
  - Can fermions condense? How fluid are superfluids?

References

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