# Dynamics of Digits: Calculating Pi with Galperin Billiards

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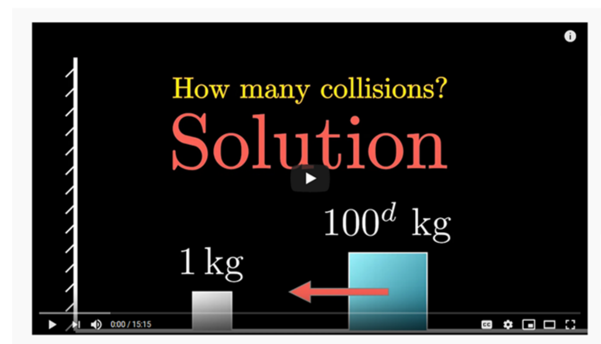
# IF YOU WATCH THREE VIDEOS ABOUT GALPERIN'S BILLIARDS…

YouTube channel: 3Blue1Brown

https://www.youtube.com/watch?v=HEfHFsfGXjs

https://www.youtube.com/watch?v=jsYwFizhncE

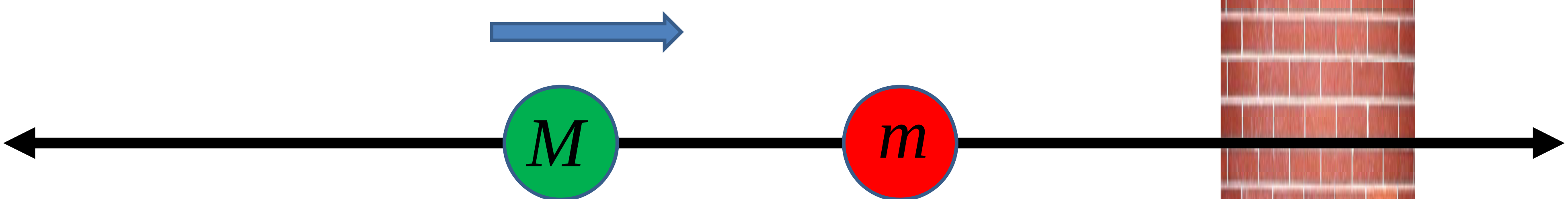
https://www.youtube.com/watch?v=brU5yLm9DZM




If you watch a fourth video…

# Galperin Billiards

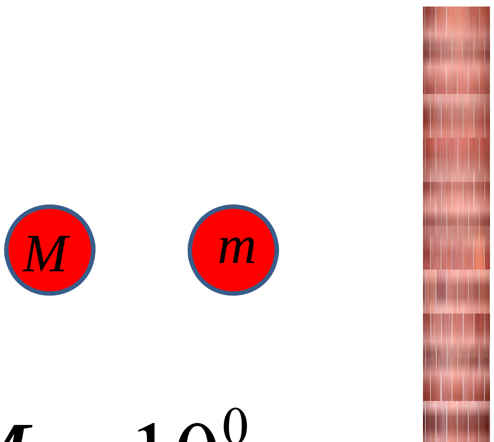
Elastic three body problem, one mass infinite




Energy is conserved, momentum is not!

Depending on mass ratio, there are one or two more conserved quantities

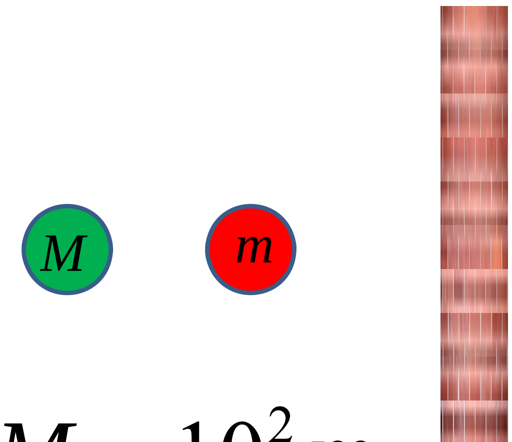
# Galperin Billiards




$$M = 10^0 m$$

number of collisions

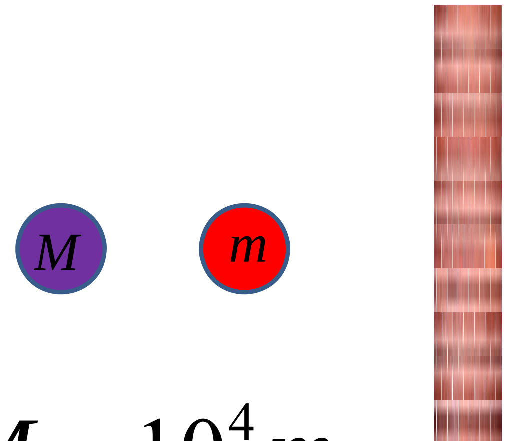
$$\Pi = 3$$




$$M = 10^2 m$$

number of collisions

$$\Pi = 31$$




$$M = 10^4 m$$

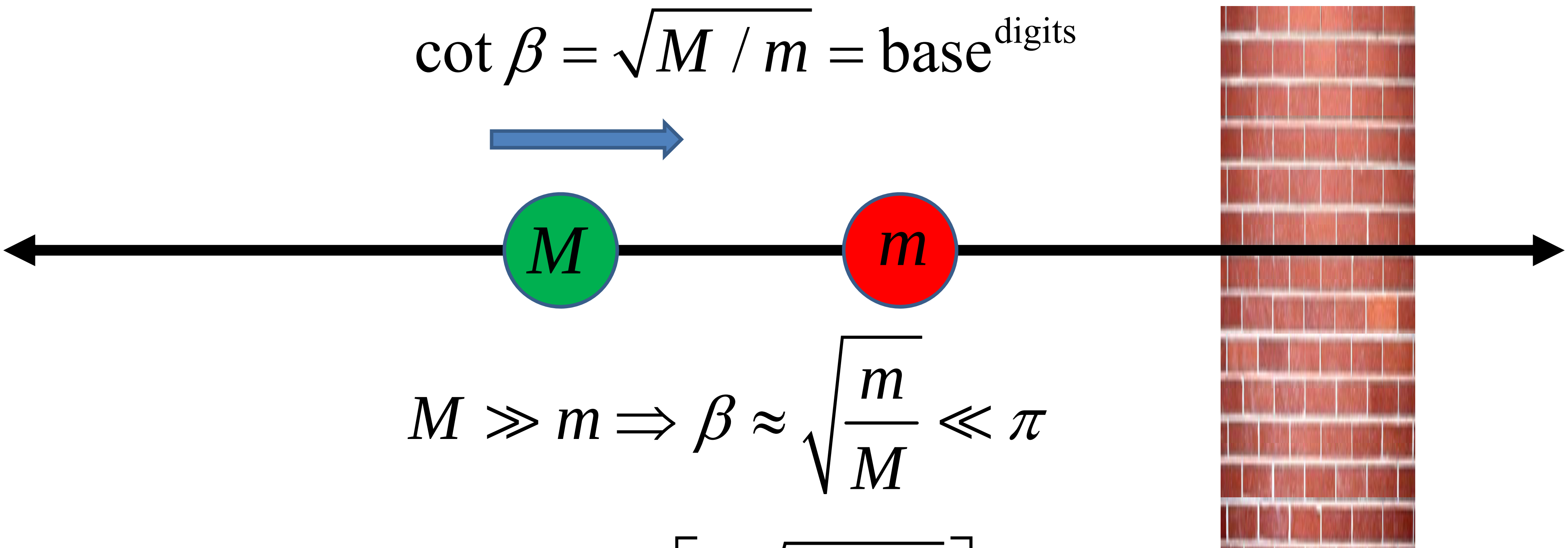
number of collisions

$$\Pi = 314$$

# Galperin Billiards

$$\Pi = \mathrm{int}\left[\pi / \beta\right]$$

$$\cot\beta = \sqrt{M / m} = \mathrm{base}^{\mathrm{digits}}$$

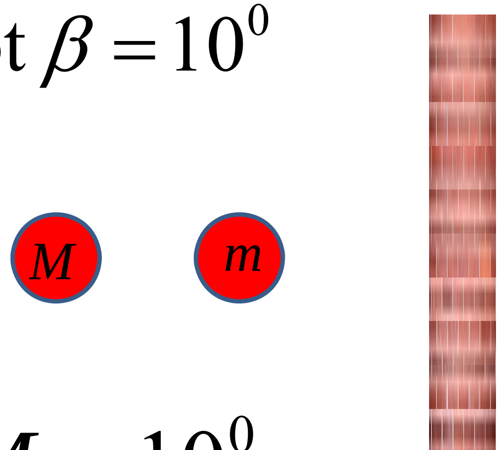



$$M \gg m \Rightarrow \beta \approx \sqrt{\frac{m}{M}} \ll \pi$$

$$\Pi \approx \mathrm{int}\left[\pi\sqrt{M / m}\right]$$

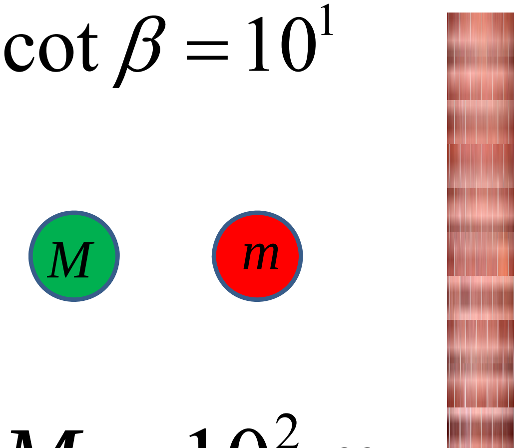
# Galperin Billiards

$$\cot\beta = 10^0$$



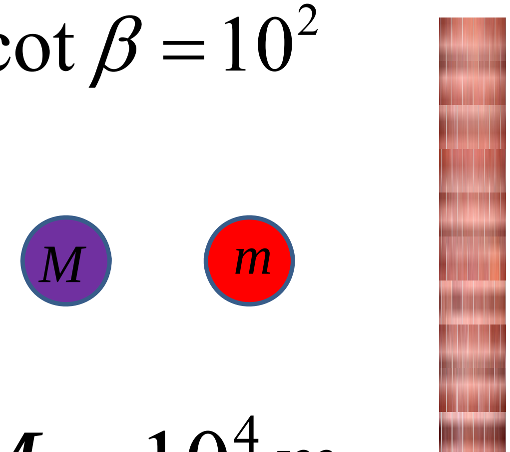

$$M = 10^0 m$$

$$\cot\beta = 10^1$$




$$M = 10^2 m$$

$$\cot\beta = 10^2$$




$$M = 10^4 m$$

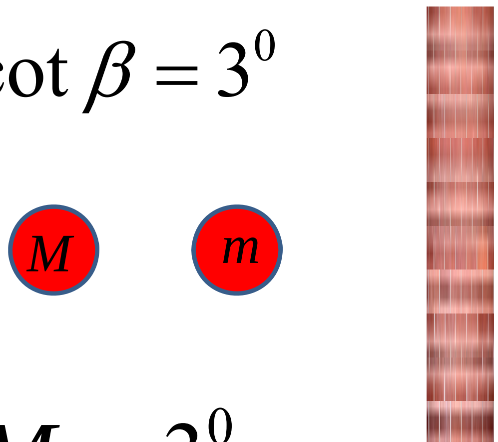
$$\Pi = 31 = \text{int}(10^1 \pi)$$

$$\Pi = 3 = \text{int}\left(10^0 \pi\right)$$

$$\Pi = 314 = \text{int}(10^2 \pi)$$

# Galperin Billiards

$$\cot\beta = 3^0$$

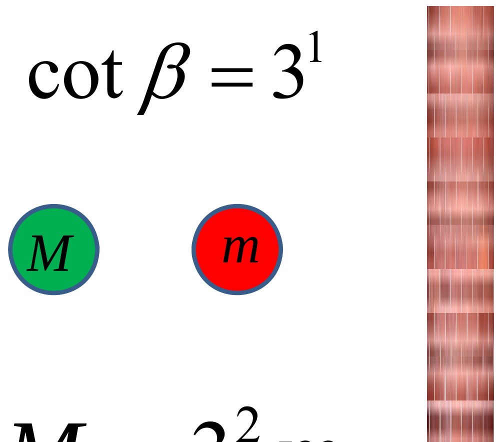



$$M = 3^0 m$$

number of collisions

$$\Pi = 3 = 10_3$$

$$\cot\beta = 3^1$$

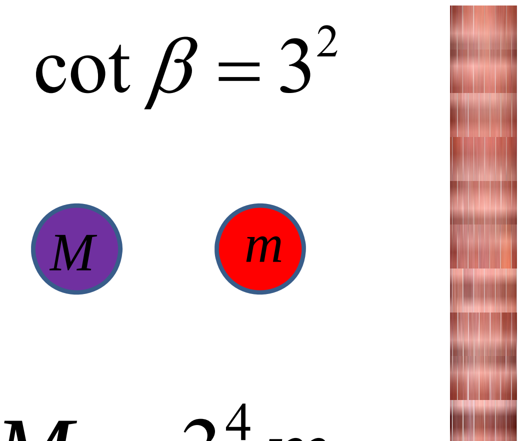



$$M = 3^2 m$$

number of collisions

$$\Pi = 9 = 100_3$$

$$\cot\beta = 3^2$$




$$M = 3^4 m$$

number of collisions

$$\Pi = 28 = 1001_3$$

$$\pi = 10.0102110_3$$

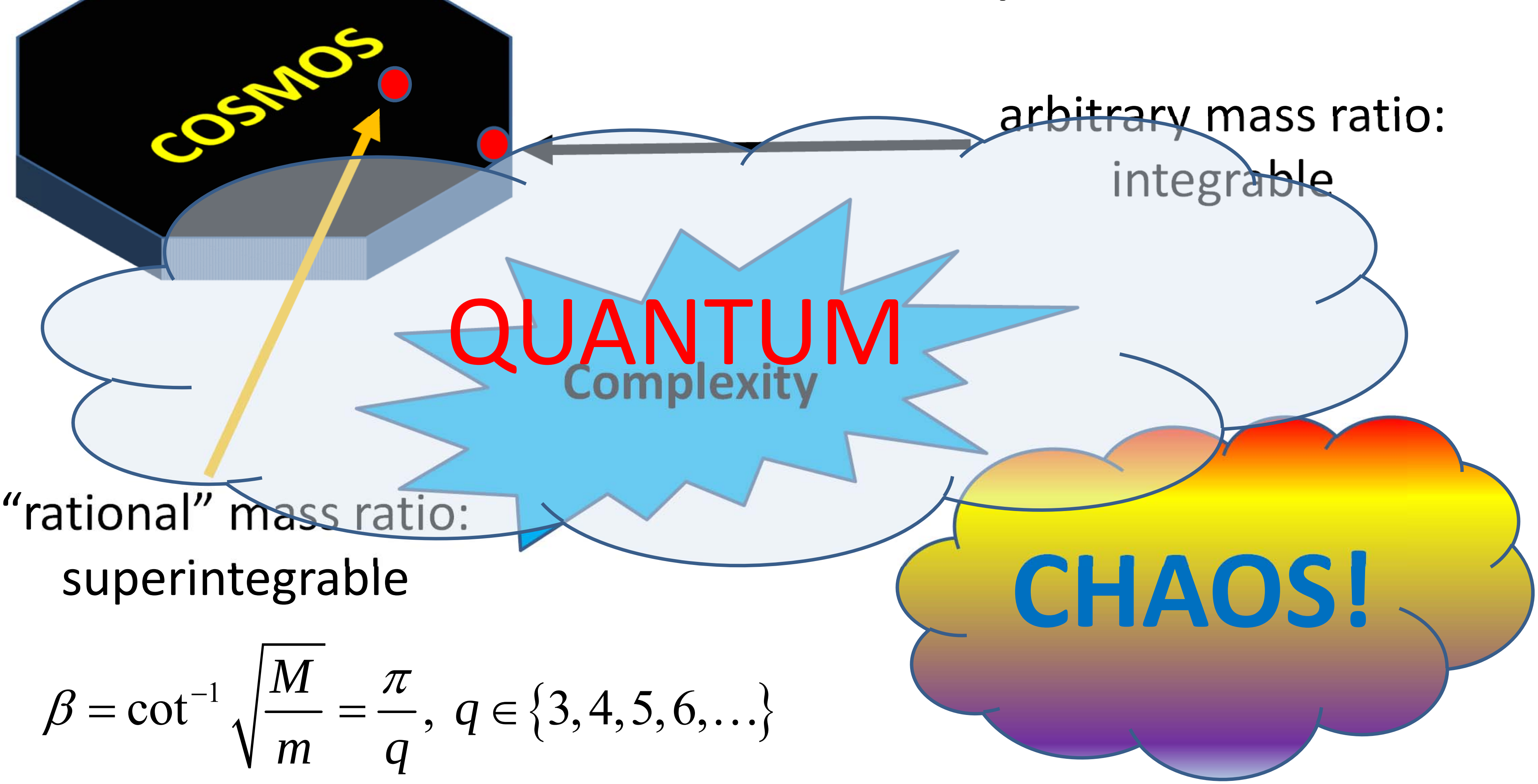
# Advertisement for a Long Paper

X. M. Aretxabaleta, M. Gonchenko, N. L. Harshman, S. G. Jackson, M. Olshanii, G. E. Astrakharchik, "The dynamics of digits: Calculating pi with Galperin's billiards", Mathematics 8, 509 (2020); arXiv:1712.06698.

- Review of previous results
  - Billiard coordinates and unfolded trajectories
- Symmetry and Dynamics
  - Pseudo angular momentum and exact results
  - Integrability and superintegrability
  - Effective Calogero interaction
- Digitization and error
  - Irrational bases

Personal Motivation

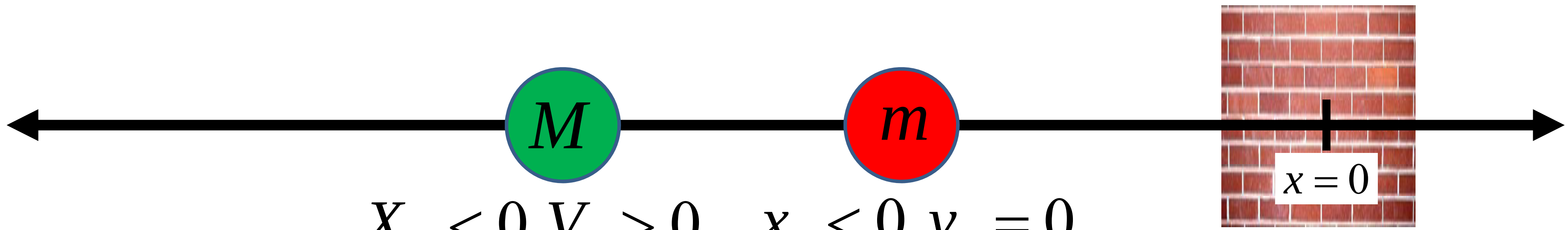
# Galperin Billiards




$$\beta = \cot^{-1}\sqrt{\frac{M}{m}} = \frac{\pi}{q},\ q \in \{3,4,5,6,\ldots\}$$

# Top four superintegrable systems

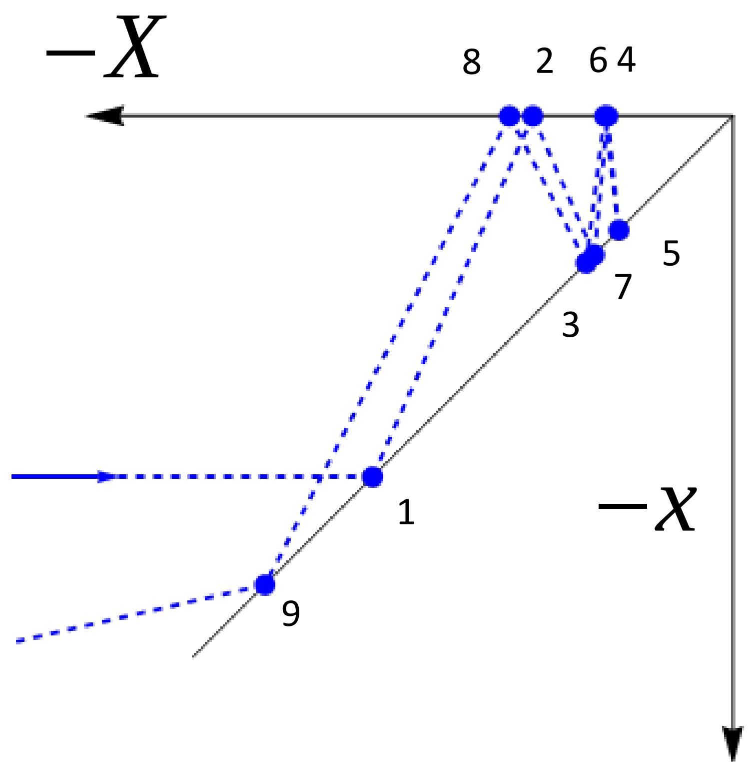
1) Kepler two-body problem
2) Particle is an isotropic harmonic trap in two or more dimensions
3) Non-interacting particles in free space or harmonic traps
4) Calogero-Sutherland-Moser Models

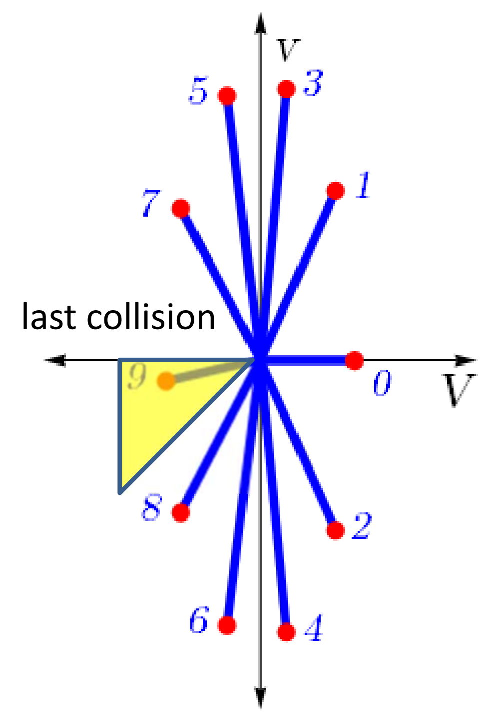


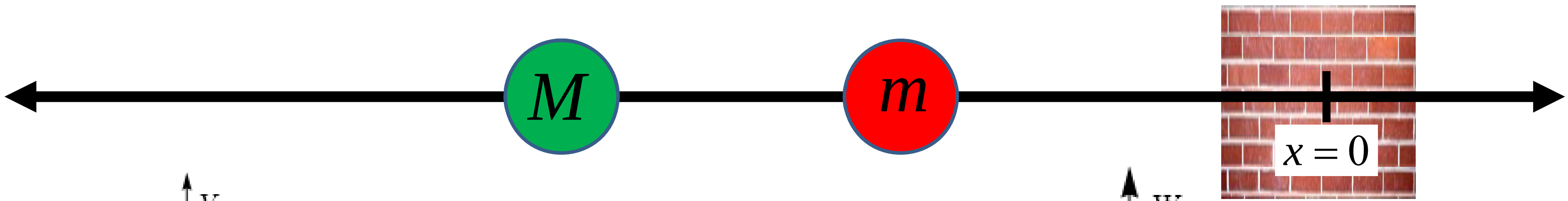

$X_0 < 0, V_0 > 0 \quad x_0 < 0, v_0 = 0$

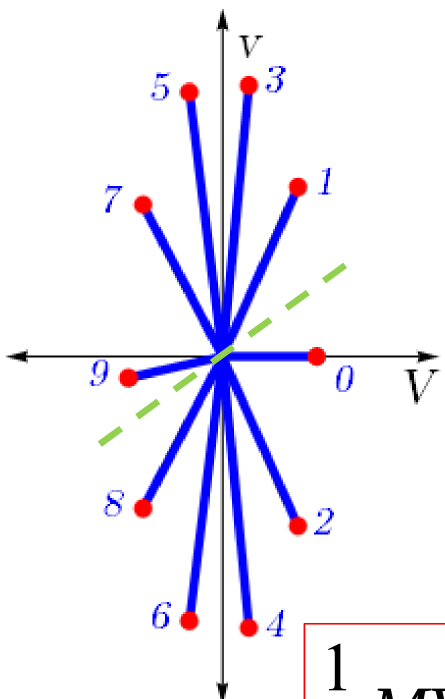
$$X_0, V_0, x_0, v_0, t_0$$
$$X_1, V_1, x_1, v_1, t_1$$
$$\vdots$$
$$X_\Pi, V_\Pi, x_\Pi, v_\Pi, t_\Pi$$

$$M / m = 9$$





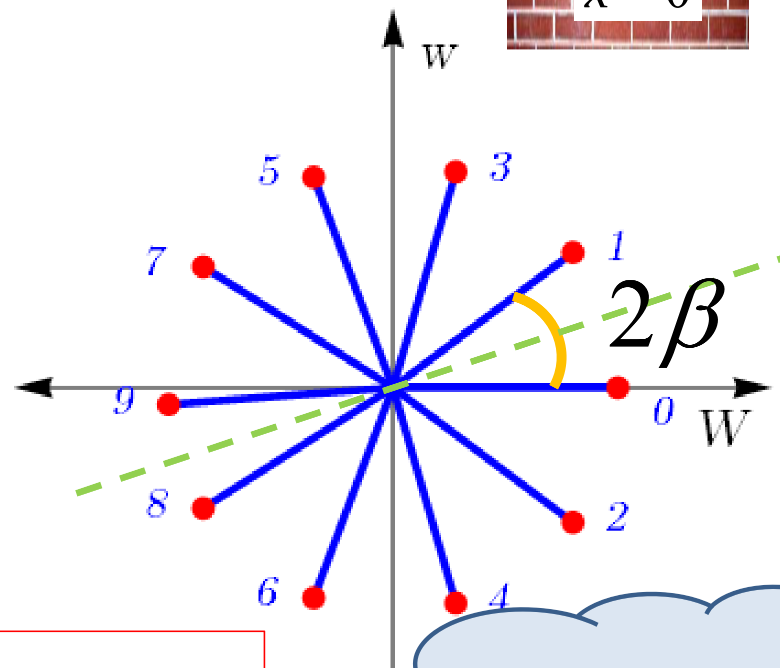




$$Y = \sqrt{M}\,X$$

$$y = \sqrt{m}\,y$$

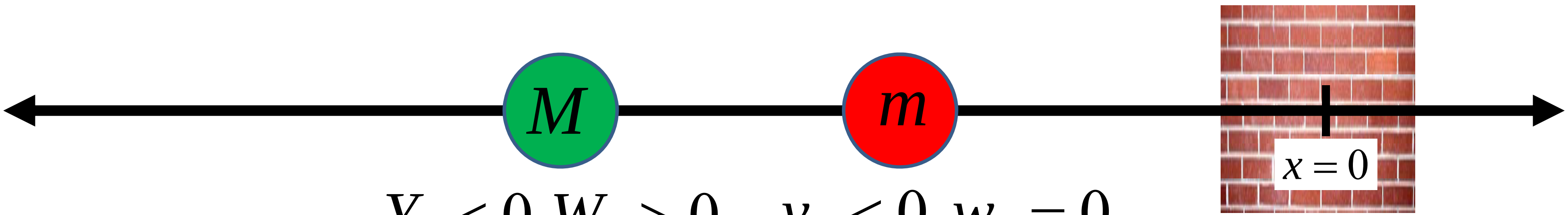
$$W = \sqrt{M}\,V$$

$$w = \sqrt{m}\,v$$

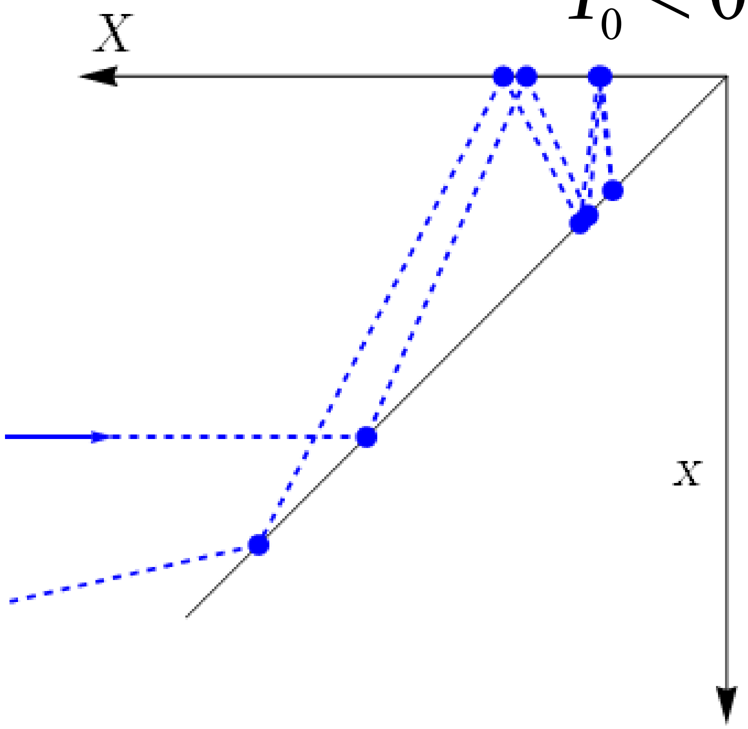



$$\frac{1}{2}MV^2 + \frac{1}{2}mv^2 = T = \frac{1}{2}\left(W^2 + w^2\right)$$

$$\tan\beta = \sqrt{m/M}$$




$$Y_0 < 0, W_0 > 0 \quad y_0 < 0, w_0 = 0$$

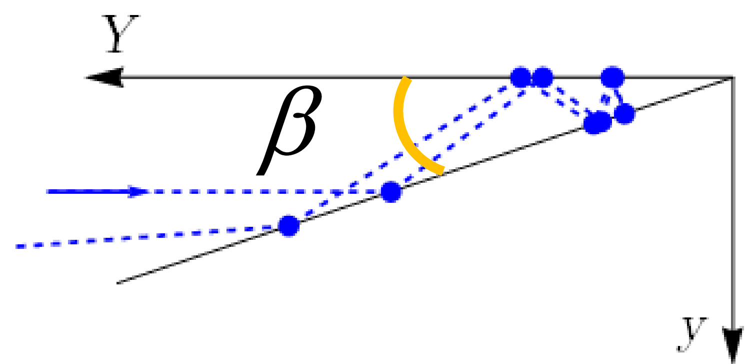



$$Y = \sqrt{M}\,X$$

$$y = \sqrt{m}\,y$$

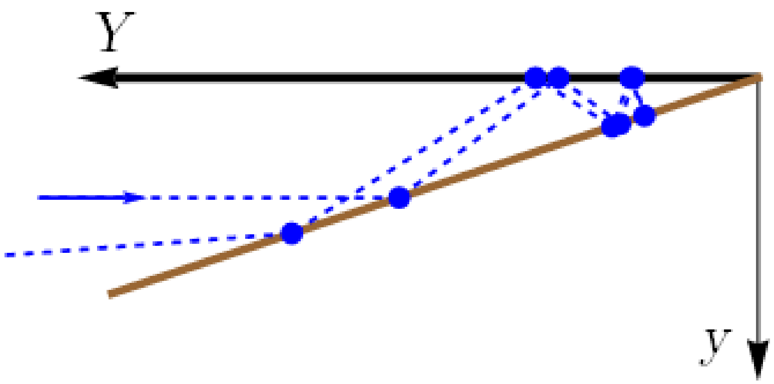
$$W = \sqrt{M}\,V$$

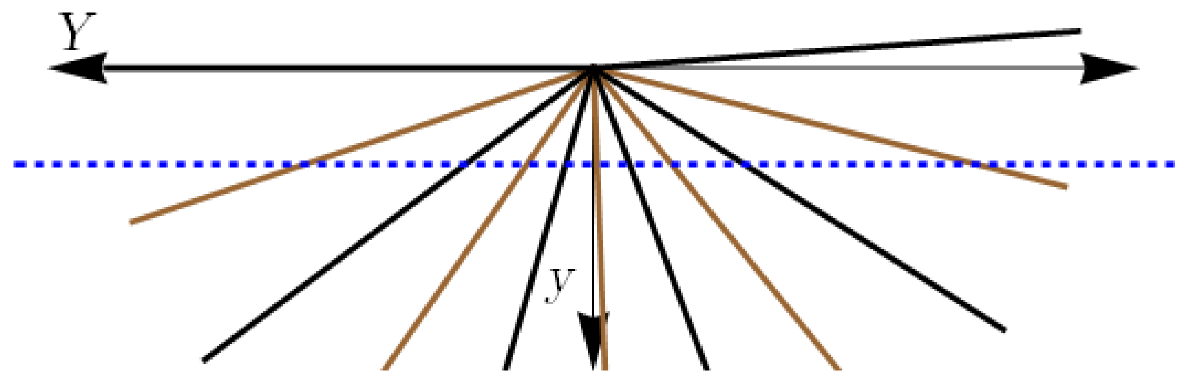
$$w = \sqrt{m}\,v$$




$$\tan\beta = \sqrt{m / M}$$

# Unfolding the trajectory




$$L^2 = (Yw - yW)^2$$

$$\boxed{L^2 = mM\left(x_0V_0\right)^2}$$

# Two degrees of freedom + Two Integrals of Motion = Integrable*

General invariants

$$T = \tfrac{1}{2}\left(W^2 + w^2\right) = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2$$

$$L^2 = (Yw - yW)^2 = mM\,(Xv - xV)^2$$

Algebra!

Pow! Pow!

Exact Results!

Initial values

$$T = \frac{1}{2}MV_0^2$$

$$L^2 = mM\left(x_0 V_0\right)^2$$

Sometimes there is a third integral of motion…

# Superintegrable mass ratios

$$\beta = \cot\sqrt{\frac{M}{m}} = \frac{\pi}{q},\ q \in \{3,4,5,6,\dots\}$$

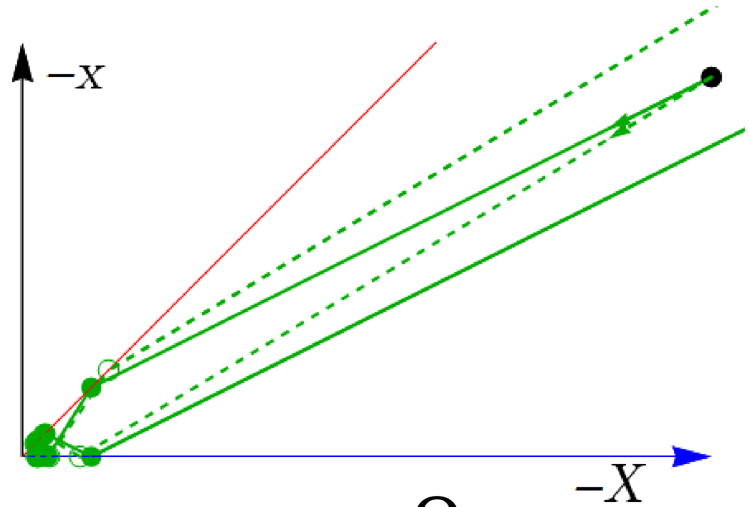
$$q=3 \quad \frac{m}{M}=3 \quad \Rightarrow \quad J = V^3 - 9Vv^2$$

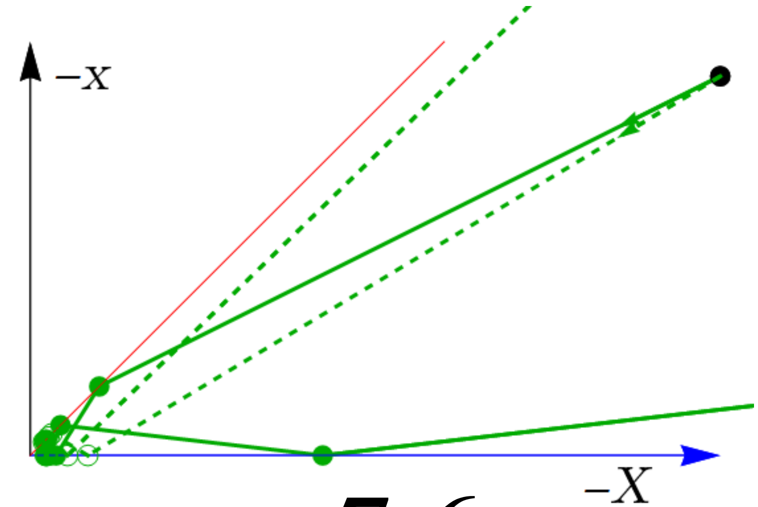
$$q=4 \quad \frac{m}{M}=1 \quad \Rightarrow \quad J = V^4 - 6V^2v^2 + v^4$$

$$q=5 \quad \frac{m}{M}=5-2\sqrt{5} \quad \Rightarrow \quad J = V^5 - 10\left(5-2\sqrt{5}\right)V^3v^2 + 25\left(9-4\sqrt{5}\right)Vv^4$$

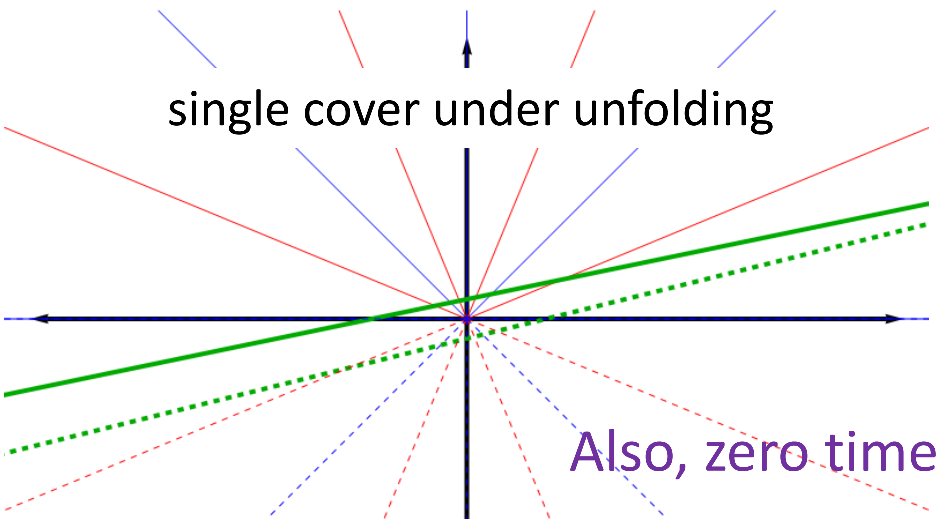
$$q=6 \quad \frac{m}{M}=\frac{1}{3} \quad \Rightarrow \quad J = V^6 - 5V^4v^2 + \frac{5}{3}V^2v^4 - \frac{1}{27}v^6$$

Consider more general initial conditions…

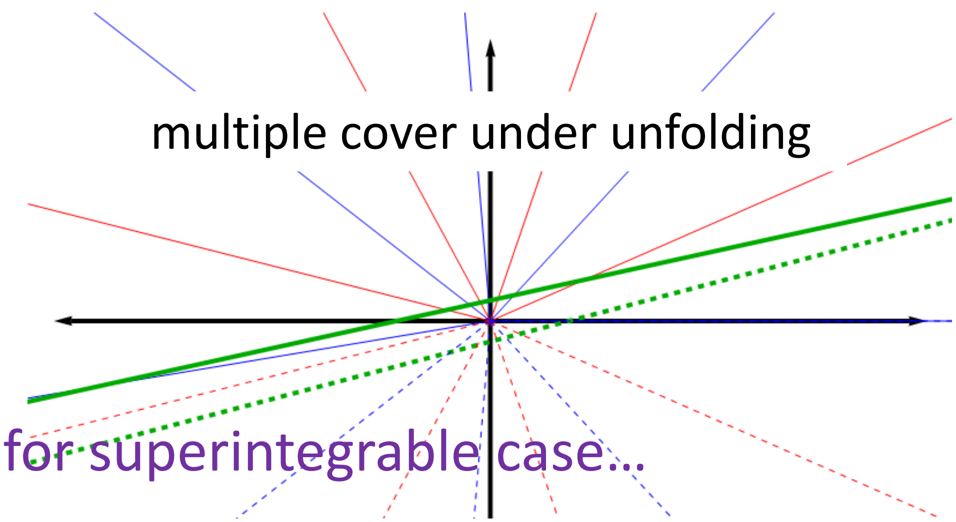



$q = 8$ $\qquad$ $q = 7.6$



single cover under unfolding



multiple cover under unfolding

Also, zero time delay for superintegrable case…

# Thanks!

- CSAAPT Organizers
- Collaborators
- AU colleagues
- Democracy

X. M. Aretxabaleta, M. Gonchenko, N. L. Harshman, S. G. Jackson, M. Olshanii, G. E. Astrakharchik, “The dynamics of digits: Calculating pi with Galperin's billiards”, Mathematics 8, 509 (2020).