Saturday 24 October 2020 CSAAPT Virtual Fall Meeting

A Bead Sliding on a Rotating Rod with Elastic and Frictional Forces

Carl E. Mungan

Physics Department, U.S. Naval Academy, Annapolis MD

mungan@usna.edu

www.usna.edu/Users/Physics/mungan

A bead slides on a rough rod.

mass of bead = m

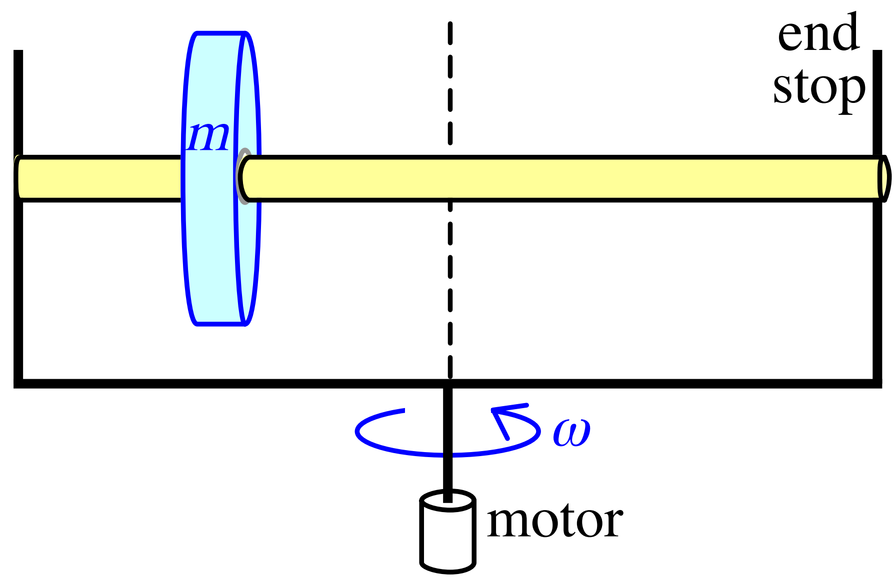
coefficient of kinetic friction = μ

The setup is in gravity-free space.

The rod rotates at a constant angular speed about its midpoint regardless of the bead's position on the rod.

angular speed of rod = ω

(can be fixed at ANY desired value)

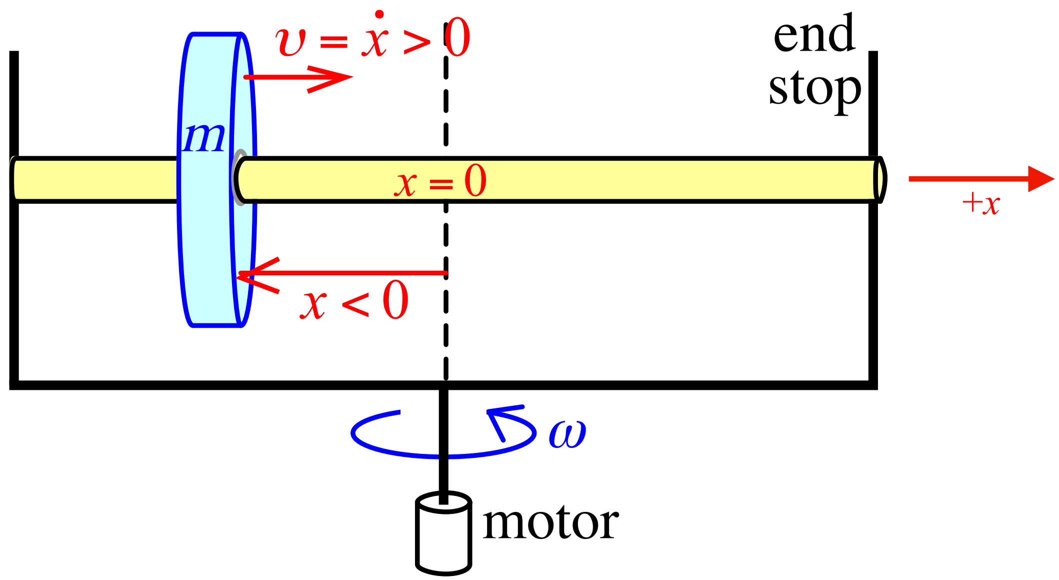


The end stops prevent the bead from sliding off the rod.

Work in the rotating reference system of the rod so that we have 1D motion along the x-axis of the rod.

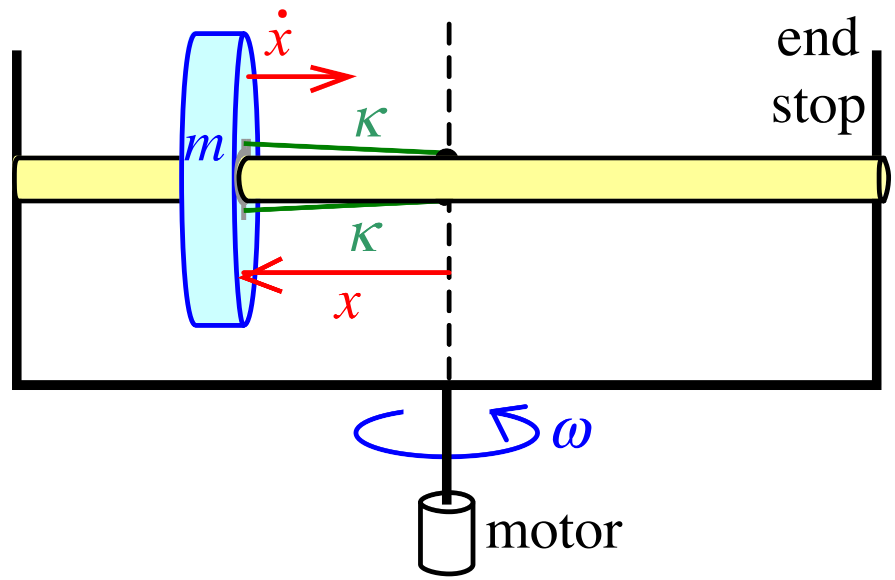
It is a noninertial reference system and so we have a centrifugal acceleration $\omega^2 x$ (equal but opposite to the centripetal acceleration) and a Coriolis acceleration $2\omega\dot{x}$ (magnitude of $2\vec{v}\times\vec{\omega}$).

(overdots denote time derivatives)



Finally a pair of ideal elastic bands (massless, Hookean, and zero relaxed length) pull the bead back toward the origin.

spring constant of each elastic band = κ



Small slits on the bead allow the elastic bands to pass through as the bead moves from one side of the rod to the other.

Suppose the bead starts at the origin $x_0 = 0$
and is given an impulsive radially outward push $\dot{x}_0 > 0$.

What are the possible motions of the bead
for different angular speeds ω of the rod?

There are 3 possibilities.

Possibility #1

At low angular speeds, the bead eventually returns to the origin with under/critical/overdamping.

The damping is due to the kinetic friction.

The friction arises because there is a normal force due to the Coriolis force.

(Remember that there is no gravity.)

Possibility #2

At high angular speeds, the bead slides outward until it hits the end stops arbitrarily far away due to the centrifugal force.

Possibility #3

At a specific intermediate crossover angular speed,
the bead moves away from the origin
to a new equilibrium position.

Is that new equilibrium stable, unstable, or neutral?

Mathematical analysis using Newton's second law:

$$m\ddot{x} = -2\kappa x + \underbrace{m\omega^2 x} - \mu \underbrace{2m\omega\dot{x}}$$

restoring force

two elastic bands

outward centrifugal force

opposes the velocity $\dot{x}$

normal force is the Coriolis force

There is no static friction because the normal force equals zero whenever the velocity is zero.

Collect the two x terms by defining $\omega_0^2 \equiv \frac{2\kappa}{m} - \omega^2$

and define the damping coefficient $\beta \equiv \mu\omega$

$$\therefore \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

This is the standard linearly damped oscillator equation.

The motion is underdamped for $\beta < \omega_0$

$$\Rightarrow \mu^2\omega^2 < \frac{2\kappa}{m} - \omega^2$$

$$\Rightarrow \omega < \sqrt{\frac{2\kappa}{m(1+\mu^2)}} \equiv \omega_c^*$$

"critical" angular speed

whereas it is critically damped for $\omega = \omega_c^*$

and it is overdamped for $\omega > \omega_c^*$

"crossover" angular speed

but only up to some angular speed ω_c.

Eventually ω gets large enough that

$$\omega_0^2 \equiv \frac{2\kappa}{m} - \omega^2 = 0$$

$$\Rightarrow \omega_c = \sqrt{\frac{2\kappa}{m}}$$

which is also the critical value in the absence of friction (when $\mu = 0$).

If $\omega > \omega_c$ then we do not get overdamped motion but instead the bead slides radially outward until it hits the end stops.

At the crossover value $\omega = \omega_c$ we have

$$m\ddot{x} = -\mu 2m\omega\dot{x}$$

$$\Rightarrow \frac{d\dot{x}}{dt} = -2\beta_c\dot{x} \text{ where } \beta_c \equiv \mu\omega_c$$

$$\therefore \boxed{\dot{x} = \dot{x}_0 e^{-2\beta_c t}}$$

so that the bead slides outward to rest at a new equilibrium position (which is neutral because the elastic and centrifugal forces balance at ANY value of x at this angular speed).

Integrate again as

$$\frac{dx}{dt} = \dot{x}_0 e^{-2\beta_\mathrm{c} t}$$

$$\Rightarrow \boxed{x = x_\infty \left(1 - e^{-2\beta_\mathrm{c} t}\right)}$$

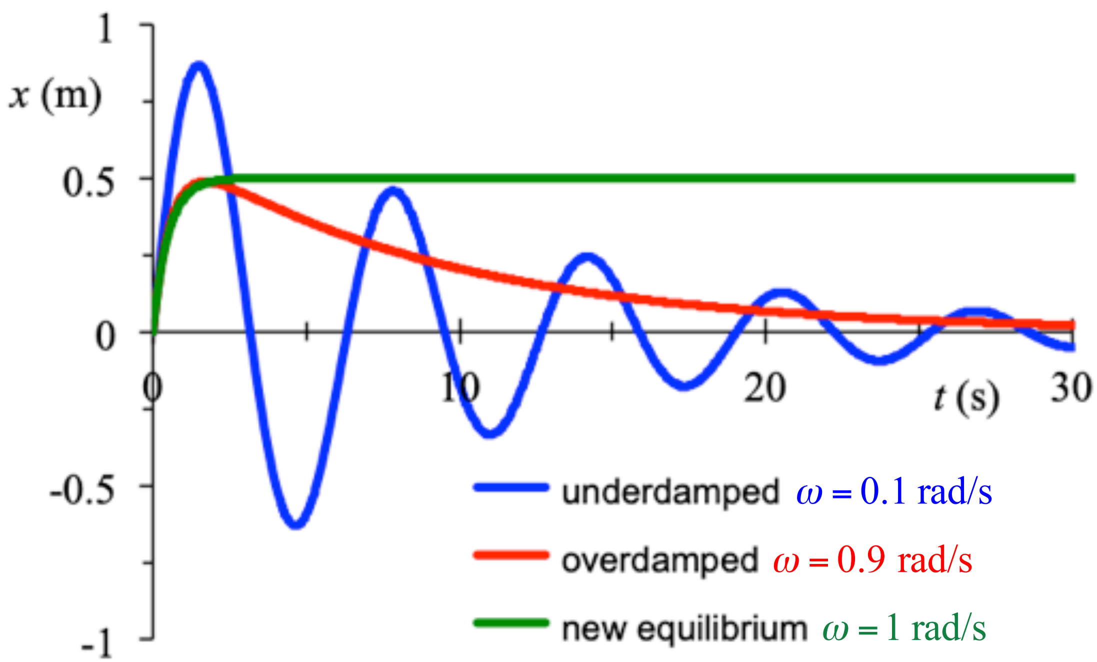
where $x_\infty \equiv \dfrac{\dot{x}_0}{2\mu}\sqrt{\dfrac{m}{2\kappa}}$

is the new equilibrium position.

As expected intuitively, x_∞ is:

- larger for bigger $\dot{x}_0$ or m;
- smaller for bigger μ or κ.

Graph of the displacement of the bead as a function of time
for three values of the angular speed of the rod:



Parameters: $m = 1$ kg, $2\kappa = 1$ N/m, $\mu = 1$, and $\dot{x}_0 = 1$ m/s $\Rightarrow \omega_C^* = 0.71$ rad/s and $\omega_C = 1$ rad/s.

REFERENCES

C.E. Mungan, "A bead sliding along a rotating rod subject to elastic and frictional forces," Phys. Teach. **58**, 532 (Nov. 2020)

which is a comment on

D.M. Djokić, "Dry friction camouflaged in viscous drag," Phys. Teach. **58**, 340 (May 2020).