A SIMPLE APPROACH TO GRAVITATIONAL ORBITS

Carl E. Mungan

Physics Department, U.S. Naval Academy, Annapolis MD

University of Virginia

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The Kepler Problem

K1L says that the bound orbits of a satellite (say a planet of mass m)
around a massive astronomical body (say a sun of mass M) are ellipses.

Great! Now how does one find the exact orbit for some planet and sun?

What minimum additional data do you need,
and then how do you use that data to construct the orbit?

Goal of this talk:
Be able to answer these two questions without notes
if a student came by your office hours tomorrow.

Initial question for discussion:

Why aren't planetary orbits circular?
After all, gravity is a central force.

Wikipedia definition:
A *central force* on a planet has
magnitude that only depends on the distance from the planet to the sun
and is *directed* toward the sun.

Suggested answer:

There are not one but TWO independent constants of the motion,
namely mechanical energy E and angular motion L.

Thus we cannot in general get a circle because it is described by a *single* constant radius.

Instead we must distort the circle in such a way that it depends on *two* independent constants.

We can choose different pairs of constants.
Let's simply choose the two maximally distorted radii,
namely perihelion (nearest) distance r_p from the planet to the sun
and aphelion (farthest) distance r_a.

How do we know there are no other independent constants of the motion?

answer: We prove the orbit is uniquely specified by constructing it!

Put the sun M at the origin,
choose the x axis to lie along the apses,
and define the orbit to be in the xy plane.

We can eliminate the planet's mass m from the problem
by using the *specific* energy $\varepsilon = E/m$ and angular momentum $l = L/m$.
Also the gravitational constant is always multiplied by the sun's mass as GM.

So the orbit is fully defined by the three values GM, ε, and l .

AT THE APSES

the planet's velocity υ and position r are perpendicular vectors so that:

$$L = rm\upsilon \quad \Rightarrow \quad \upsilon = \frac{\ell}{r} \qquad (1)$$

$$\varepsilon = \frac{1}{2}\upsilon^2 - \frac{GM}{r} \qquad (2)$$

substitute Eq. (1) into (2) and rearrange to get:

$$\boxed{\varepsilon r^2 + GMr - \ell^2/2 = 0} \qquad (3)$$

special cases:

$\ell = 0 \Rightarrow$ radial (degenerate) orbit from $r = 0$ to $\sqrt{GM/\varepsilon}$

$\varepsilon = 0 \Rightarrow$ parabolic (escape) orbit from $r = \ell^2/2GM$ to ∞

$$\varepsilon \neq 0 \;\Rightarrow\; r = \frac{-GM \pm \sqrt{(GM)^2 + 2\varepsilon\ell^2}}{2\varepsilon} \qquad (4)$$

for $\varepsilon > 0$: choose the + sign to get the unique real solution r_p for a hyperbolic orbit

for $\varepsilon < 0$: the square root is zero if ℓ is $\ell_\text{c} = GM\big/\sqrt{2|\varepsilon|}$ in which case the orbit is circular ($r_\text{p} = r_\text{a}$)

The elliptical orbit for $\varepsilon < 0$ and $0 < \ell < \ell_\text{c}$ has r_p and r_a given by Eq. (4).

(The proof by construction is now complete.)

Given the two apses, we can find other elliptical parameters:

semimajor length $a = (r_\mathrm{p} + r_\mathrm{a})/2$

eccentricity $e = 1 - r_\mathrm{p}/a$

semiminor length $b = a\sqrt{1 - e^2}$

To draw the ellipse in Excel, shift the origin to the geometric center:

$$X \equiv x + ae \text{ and } Y \equiv y \quad \Rightarrow \quad \left(\frac{X}{a}\right)^2 + \left(\frac{Y}{b}\right)^2 = 1$$

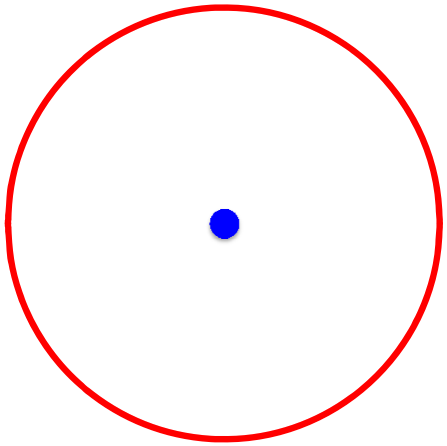
As a starting point, draw a circular orbit of radius r_c
in which case the angular momentum is l_c and the energy is:

$$\varepsilon_c = -\frac{GM}{2r_c}$$

<switch to Excel>

G	6.67E-11
M	5.98E+24
GM	3.99E+14
R	6.37E+06
r_c	6.37E+07
E_c	-3.13E+06
L_c	1.59E+11
L	1.59E+11
radical	0.00E+00
sqrt	0.00E+00
r_p	6.37E+07
r_a	6.37E+07
a	6.37E+07
e	0.000
b	6.37E+07
N	100
dX	6.37E+05
f	
0.00E+00	0

X	x	positive Y	minus Y
6.37E+07	6.37E+07	0.00E+00	0.00E+00
6.31E+07	6.31E+07	8.99E+06	-8.99E+06
6.24E+07	6.24E+07	1.27E+07	-1.27E+07
6.18E+07	6.18E+07	1.55E+07	-1.55E+07
6.12E+07	6.12E+07	1.78E+07	-1.78E+07
6.05E+07	6.05E+07	1.99E+07	-1.99E+07
5.99E+07	5.99E+07	2.17E+07	-2.17E+07
5.92E+07	5.92E+07	2.34E+07	-2.34E+07
5.86E+07	5.86E+07	2.50E+07	-2.50E+07
5.80E+07	5.80E+07	2.64E+07	-2.64E+07
5.73E+07	5.73E+07	2.78E+07	-2.78E+07
5.67E+07	5.67E+07	2.90E+07	-2.90E+07
5.61E+07	5.61E+07	3.03E+07	-3.03E+07
5.54E+07	5.54E+07	3.14E+07	-3.14E+07
5.48E+07	5.48E+07	3.25E+07	-3.25E+07
5.41E+07	5.41E+07	3.36E+07	-3.36E+07
5.35E+07	5.35E+07	3.46E+07	-3.46E+07
5.29E+07	5.29E+07	3.55E+07	-3.55E+07
5.22E+07	5.22E+07	3.65E+07	-3.65E+07
5.16E+07	5.16E+07	3.74E+07	-3.74E+07
5.10E+07	5.10E+07	3.82E+07	-3.82E+07
5.03E+07	5.03E+07	3.91E+07	-3.91E+07
4.97E+07	4.97E+07	3.99E+07	-3.99E+07
4.90E+07	4.90E+07	4.06E+07	-4.06E+07
4.84E+07	4.84E+07	4.14E+07	-4.14E+07
4.78E+07	4.78E+07	4.21E+07	-4.21E+07
4.71E+07	4.71E+07	4.28E+07	-4.28E+07
4.65E+07	4.65E+07	4.35E+07	-4.35E+07
4.59E+07	4.59E+07	4.42E+07	-4.42E+07
4.52E+07	4.52E+07	4.49E+07	-4.49E+07
4.46E+07	4.46E+07	4.55E+07	-4.55E+07
4.40E+07	4.40E+07	4.61E+07	-4.61E+07
4.33E+07	4.33E+07	4.67E+07	-4.67E+07
4.27E+07	4.27E+07	4.73E+07	-4.73E+07
4.20E+07	4.20E+07	4.79E+07	-4.79E+07
4.14E+07	4.14E+07	4.84E+07	-4.84E+07
4.08E+07	4.08E+07	4.89E+07	-4.89E+07
4.01E+07	4.01E+07	4.95E+07	-4.95E+07
3.95E+07	3.95E+07	5.00E+07	-5.00E+07
3.89E+07	3.89E+07	5.05E+07	-5.05E+07
3.82E+07	3.82E+07	5.10E+07	-5.10E+07
3.76E+07	3.76E+07	5.14E+07	-5.14E+07
3.69E+07	3.69E+07	5.19E+07	-5.19E+07
3.63E+07	3.63E+07	5.23E+07	-5.23E+07
3.57E+07	3.57E+07	5.28E+07	-5.28E+07
3.50E+07	3.50E+07	5.32E+07	-5.32E+07
3.44E+07	3.44E+07	5.36E+07	-5.36E+07



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r_p		2.85E+07
r_a		9.89E+07
a		6.37E+07
e		0.553
b		5.31E+07
N		100
dX		6.37E+05
f		
	-3.52E+07	0

X	x	positive Y	minus Y
6.37E+07	2.85E+07	0.00E+00	0.00E+00
6.31E+07	2.79E+07	7.49E+06	-7.49E+06
6.24E+07	2.72E+07	1.06E+07	-1.06E+07
6.18E+07	2.66E+07	1.29E+07	-1.29E+07
6.12E+07	2.59E+07	1.49E+07	-1.49E+07
6.05E+07	2.53E+07	1.66E+07	-1.66E+07
5.99E+07	2.47E+07	1.81E+07	-1.81E+07
5.92E+07	2.40E+07	1.95E+07	-1.95E+07
5.86E+07	2.34E+07	2.08E+07	-2.08E+07
5.80E+07	2.28E+07	2.20E+07	-2.20E+07
5.73E+07	2.21E+07	2.31E+07	-2.31E+07
5.67E+07	2.15E+07	2.42E+07	-2.42E+07
5.61E+07	2.08E+07	2.52E+07	-2.52E+07
5.54E+07	2.02E+07	2.62E+07	-2.62E+07
5.48E+07	1.96E+07	2.71E+07	-2.71E+07
5.41E+07	1.89E+07	2.80E+07	-2.80E+07
5.35E+07	1.83E+07	2.88E+07	-2.88E+07
5.29E+07	1.77E+07	2.96E+07	-2.96E+07
5.22E+07	1.70E+07	3.04E+07	-3.04E+07
5.16E+07	1.64E+07	3.11E+07	-3.11E+07
5.10E+07	1.57E+07	3.19E+07	-3.19E+07
5.03E+07	1.51E+07	3.25E+07	-3.25E+07
4.97E+07	1.45E+07	3.32E+07	-3.32E+07
4.90E+07	1.38E+07	3.39E+07	-3.39E+07
4.84E+07	1.32E+07	3.45E+07	-3.45E+07
4.78E+07	1.26E+07	3.51E+07	-3.51E+07
4.71E+07	1.19E+07	3.57E+07	-3.57E+07
4.65E+07	1.13E+07	3.63E+07	-3.63E+07
4.59E+07	1.07E+07	3.68E+07	-3.68E+07
4.52E+07	1.00E+07	3.74E+07	-3.74E+07
4.46E+07	9.38E+06	3.79E+07	-3.79E+07
4.40E+07	8.74E+06	3.84E+07	-3.84E+07
4.33E+07	8.10E+06	3.89E+07	-3.89E+07
4.27E+07	7.47E+06	3.94E+07	-3.94E+07
4.20E+07	6.83E+06	3.99E+07	-3.99E+07
4.14E+07	6.19E+06	4.03E+07	-4.03E+07
4.08E+07	5.56E+06	4.08E+07	-4.08E+07
4.01E+07	4.92E+06	4.12E+07	-4.12E+07
3.95E+07	4.28E+06	4.16E+07	-4.16E+07
3.89E+07	3.65E+06	4.21E+07	-4.21E+07
3.82E+07	3.01E+06	4.25E+07	-4.25E+07
3.76E+07	2.37E+06	4.29E+07	-4.29E+07
3.69E+07	1.73E+06	4.32E+07	-4.32E+07
3.63E+07	1.10E+06	4.36E+07	-4.36E+07
3.57E+07	4.61E+05	4.40E+07	-4.40E+07
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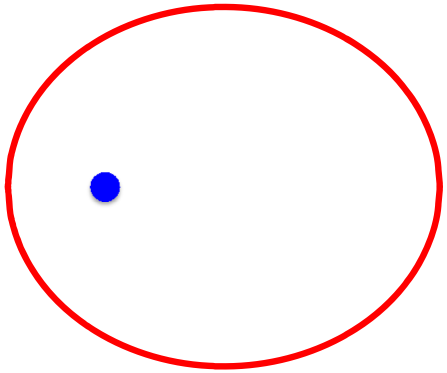

Summary

One can construct a gravitational orbit by eliminating speed between the expressions for mechanical energy and angular momentum to get a quadratic equation for the apses.

Acknowledgment

Philip Blanco, Grossmont Community College, CA