# Hidden Fractals in the Dynamics of the Compound Double Pendulum

Presented by May Palace

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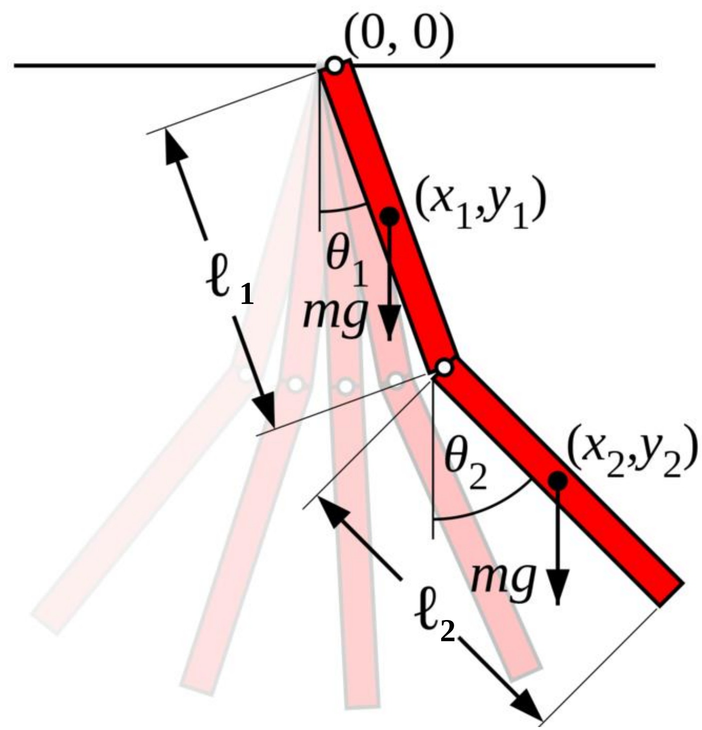
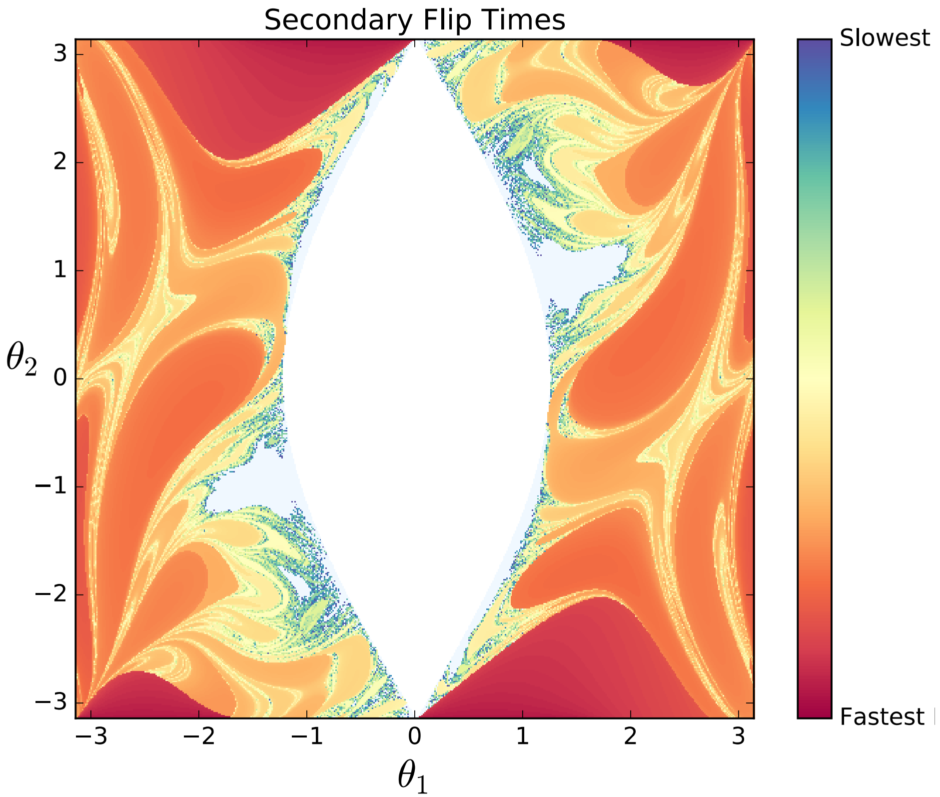
# Chaotic Dynamical Systems

## Compound double pendulum

* “When the present determines the future, but the approximate present does not approximately determine the future.” *Edward Lorenz*
* Relevant examples for fluid dynamics, ecology, economics, astrophysics, etc.
* Compound Double Pendulum
  * Rich dynamical behavior from a seemingly simple mechanical system
  * Equations of motion must be solved for numerically (Fourth-order Runge-Kutta with adaptive step sizes)



# Plotting a "Flip Portrait"




"Flip Portrait" for the first flip events of the secondary arm. Pendulum initialized with arms of equal lengths and masses.

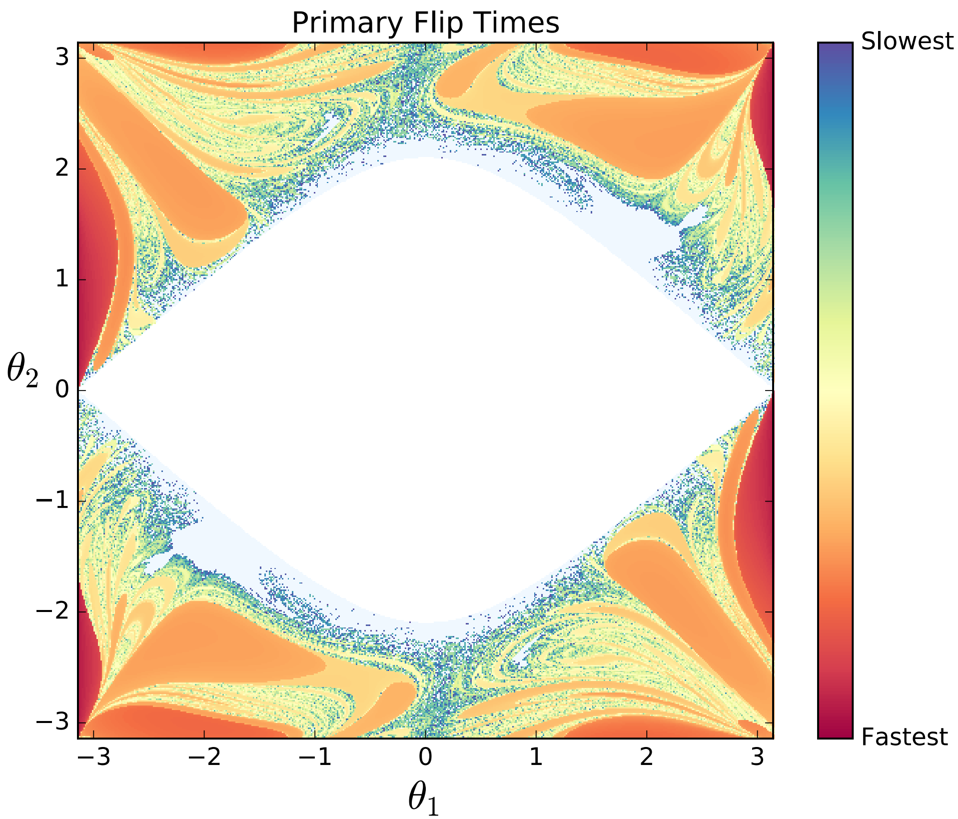
- First flip event as a function of initial conditions
  - Grid initialized from –π to +π
  - At occurrence of first flip event, point is colored according to color-map on right
- Lens shape encloses a region where it is energetically impossible for a flip to occur ("Forbidden Zone")
  - Described by initial Hamiltonian

$$H = \sum \dot{q}_i \frac{\delta L}{\delta \dot{q}_i} - L$$

$$H_\circ = -\frac{1}{2} mgl(3\cos\theta_1 + \cos\theta_2)$$

$$3\cos\theta_1 + \cos\theta_2 > 2$$

# “Flip Portrait” Examination




Compound double pendulum with 1:4 length ratio of the primary and secondary arms. The lens shape has now rotated.

- Looking for changes resulting from varying pendulum parameters
  - Pendulum arms may be equally energetically likely to flip, or the primary may become the favored arm

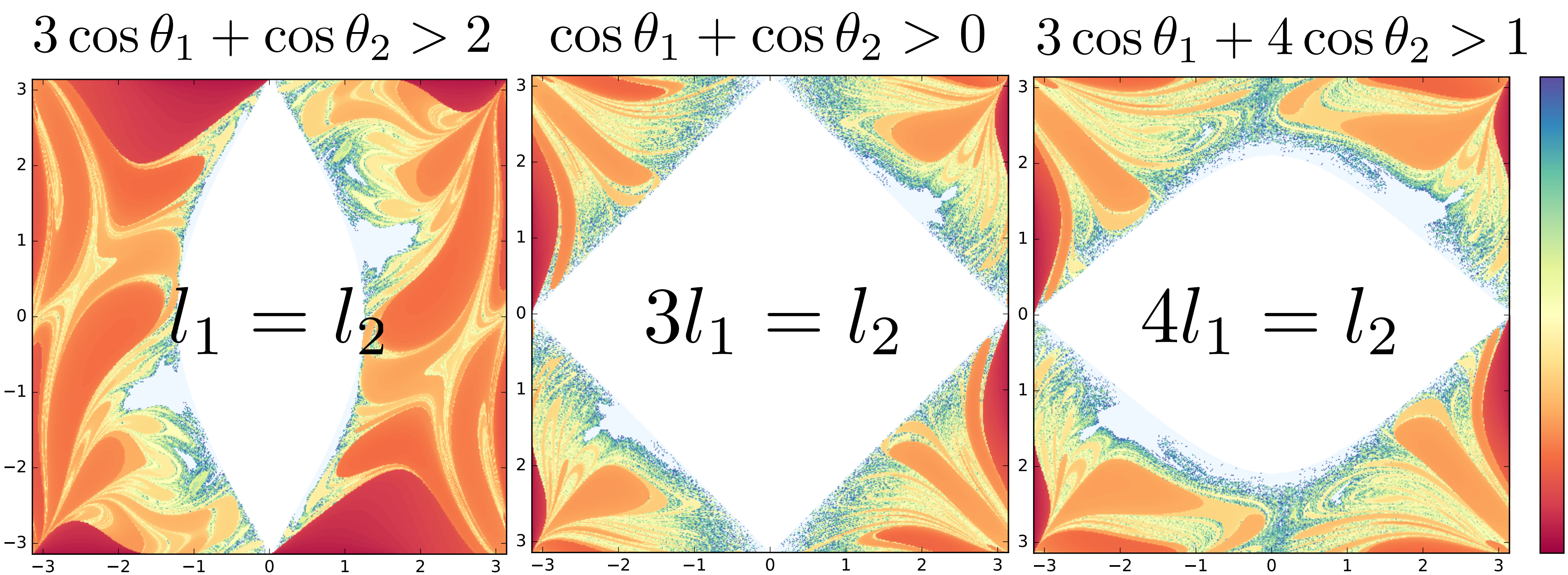
$$H_\circ = -\frac{1}{2}mg(3l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$3l_1 = l_2 \quad \cos\theta_1 + \cos\theta_2 > 0$$

$$4l_1 = l_2 \quad 3\cos\theta_1 + 4\cos\theta_2 > 1$$

# Three Cases of Pendulum Parameters

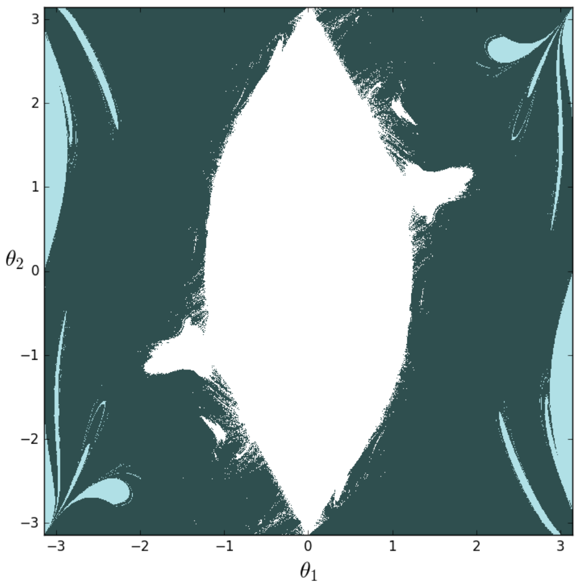
*As the length ratio increases, the forbidden zone expands horizontally and then shrinks vertically*



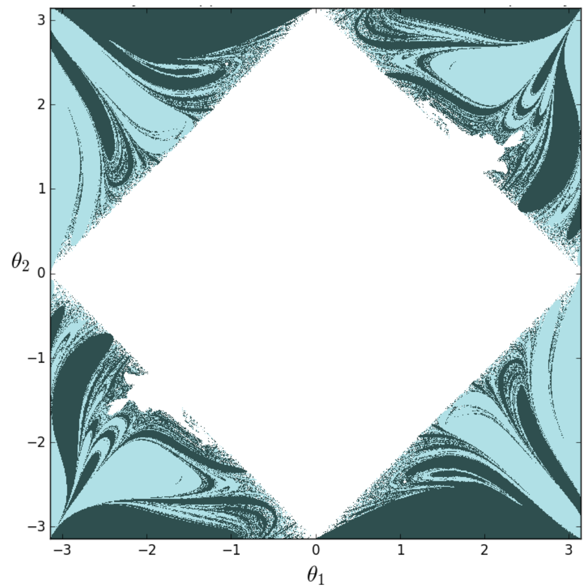

*Notice: flips overall take longer, and the "time zones" take on different shapes*

# Three Cases of Pendulum Parameters

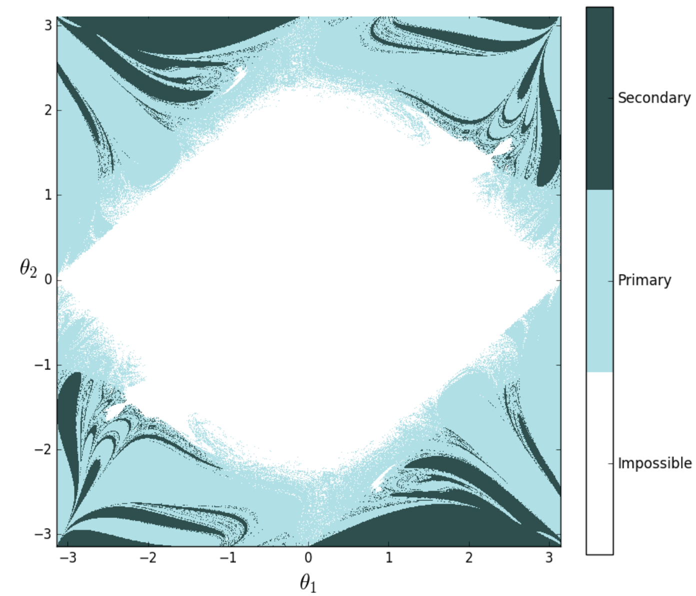
*Do the number of relative first flips that occur change as we vary favorability?*




The *secondary* arm flipped first 12.08 times more than the primary arm did when $l_1 = l_2$



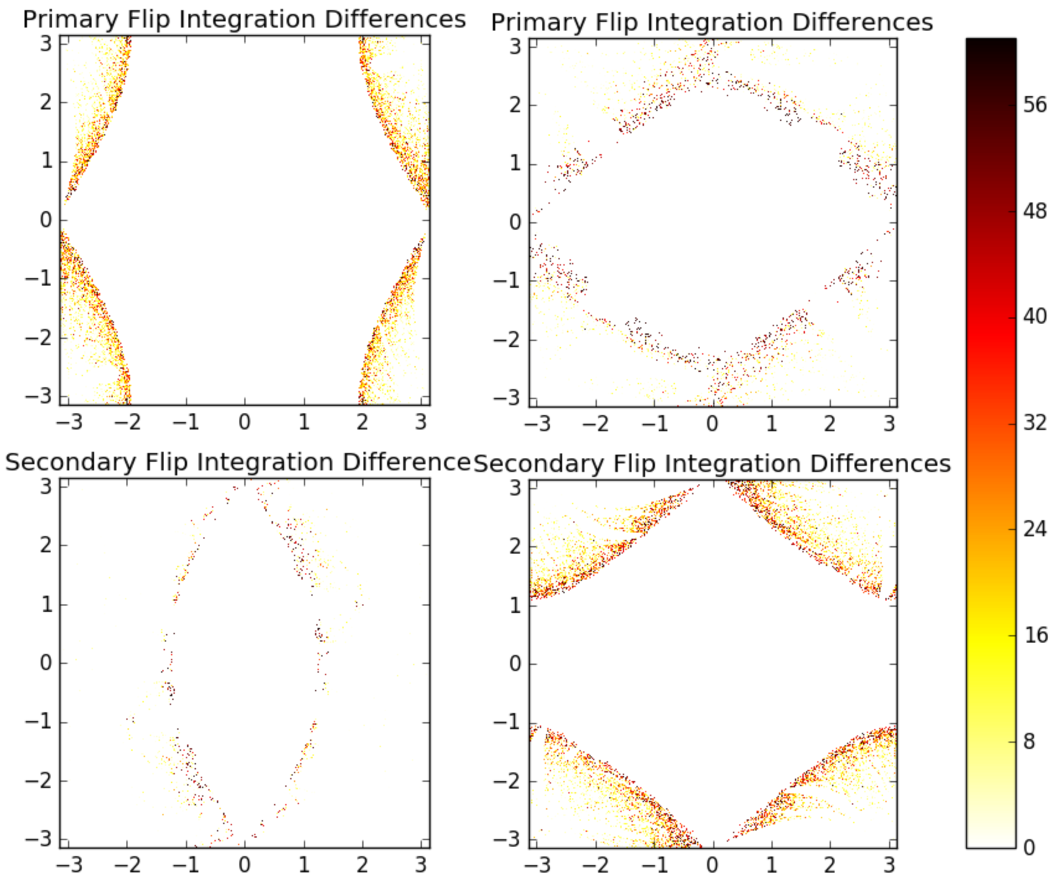

The *secondary* arm flipped first 1.11 times more than the primary arm did when $3l_1 = l_2$




The *primary* arm flipped first 2.74 times more than the secondary arm did when $4l_1 = l_2$

# Integration Tolerance

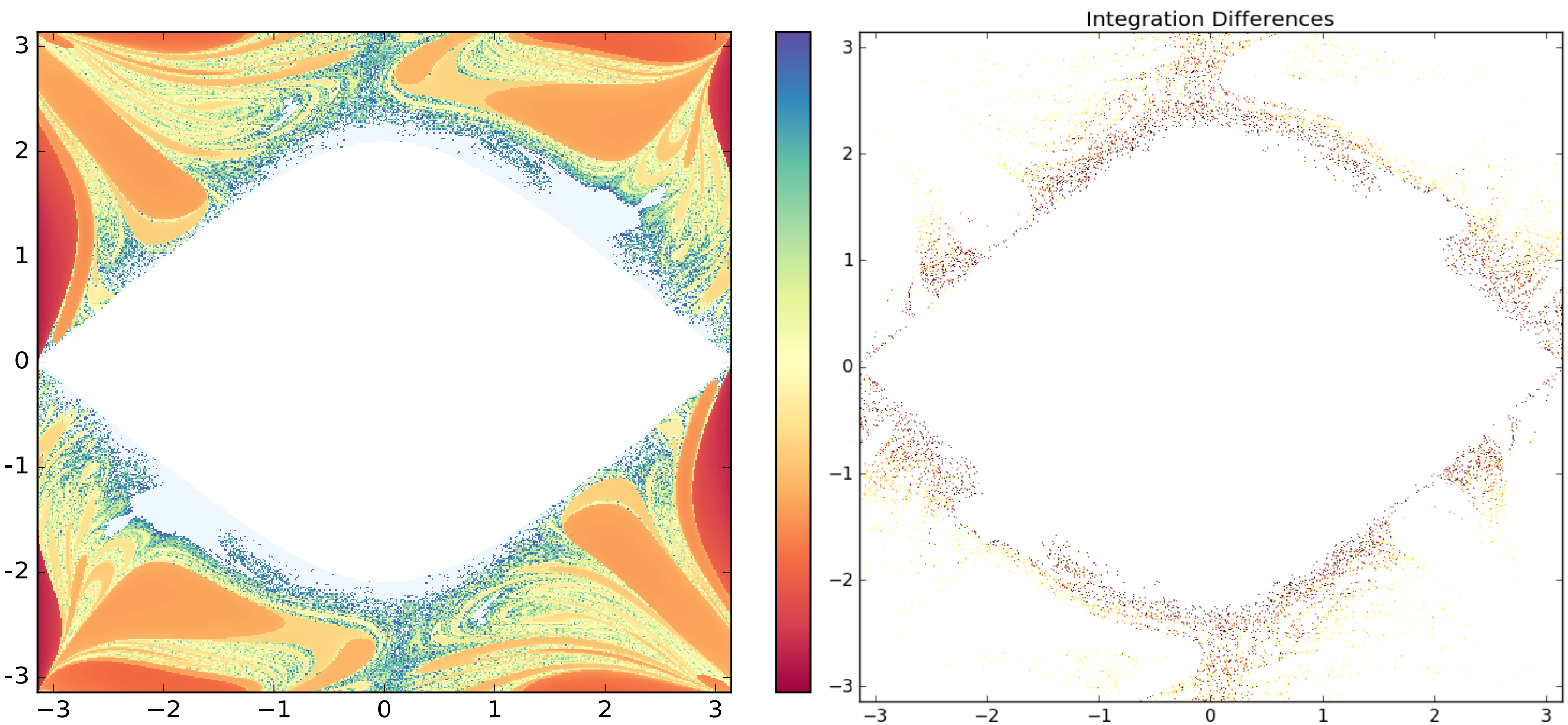
*Can round-off errors from numerical integration cause exponentially divergent behavior?*




*Left image showing equal length ratio, right is 4:1 length ratio*

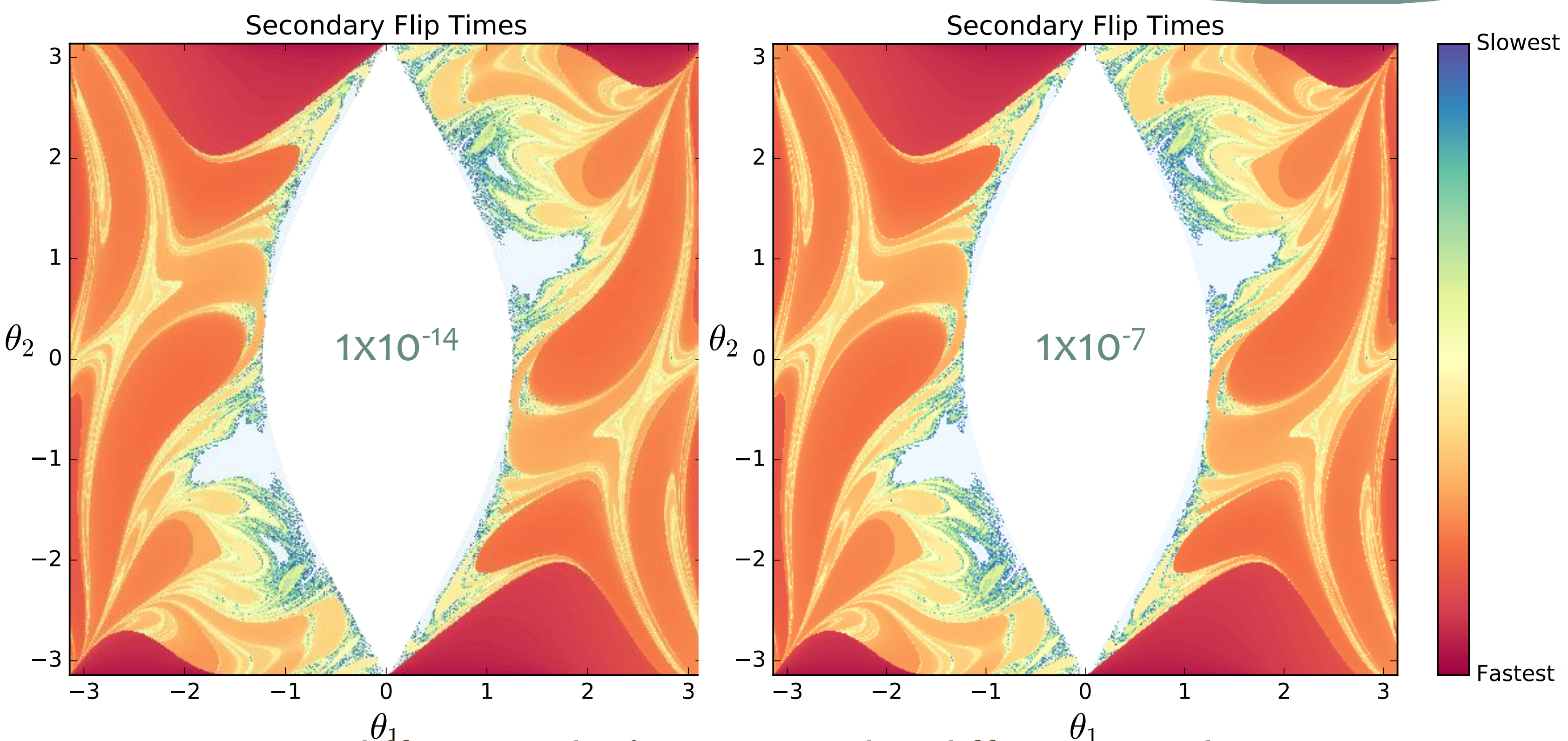
- "Typical" documented tolerance used for our integrator is $1x10^{-7}$
- Can test against $1x10^{-14}$ and plot differences (in seconds)
  - Same –π to +π grid for each picture
  - Both tolerances integrated, then subtracted and plotted to see where largest differences occur
- Then see if differences correspond to differences in the flip portraits

# Integration Tolerance




*Longest flips correspond to largest differences*

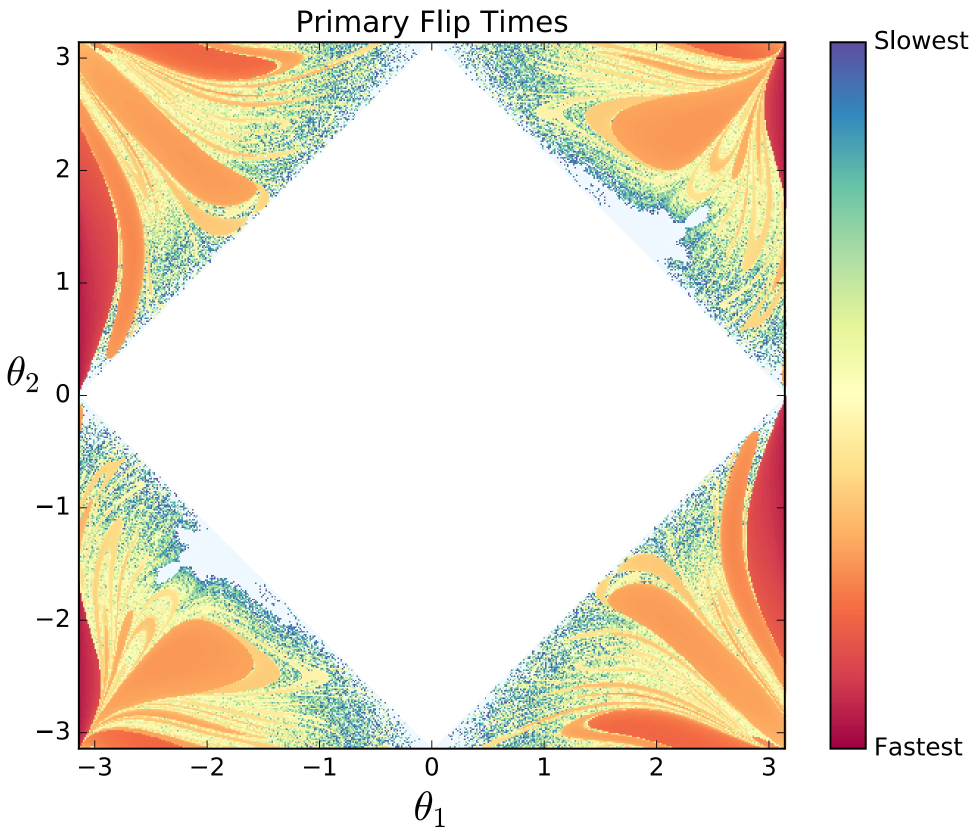
# *Integration Tolerance*




*Integration differences don't correspond to differences in the pictures*

# Further Research

* Improve resolution
  * Currently 1600x1600 points
* Investigate more length ratios to find general relations
  * Relative first flips
  * Size of time zones
  * Overall color differences
* Use different integration technique



# *References*

Strogatz, Stephen. *Nonlinear Dynamics and Chaos* Perseus Books Publishing, 1994.

J. Heyl *The Double Pendulum Fractal*, 2008.

E. Weisstein *Double Pendulum* Eric Weisstein's World of Physics, 2007.

# Supplemental

$$L = \frac{1}{6}(3g(l_1(m_1+2m_2)\cos\theta_1 + l_2 m_2 \cos\theta_2) + l_2^2(m_1+3m_2))\dot{\theta_1}^2 + 3l_1 l_2 m_2 \cos\theta_1 - \theta_2\dot{\theta_1}\dot{\theta_2} + l_2^2 m_2 \dot{\theta_2}^2)$$

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$$H_\circ = -\frac{1}{2}mgl(3\cos\theta_1 + \cos\theta_2)$$

$$\theta_1 = \pi, \theta_2 = 0 \qquad -\frac{1}{2}mgl(3\cos\theta_1 + \cos\theta_2) > -\frac{1}{2}(-3+1) = mgl$$

$$\theta_1 = 0, \theta_2 = \pi \qquad -\frac{1}{2}mgl(3\cos\theta_1 + \cos\theta_2) > -\frac{1}{2}(3-1) = -mgl$$

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$$H_\circ = -\frac{1}{2}mg(3l_1\cos\theta_1 + l_2\cos\theta_2)$$

*Similar analysis can be done with Hamiltonian for pendulum with different lengths*